Int. J. Heat Mass Transfer. Vol. 8, pp. 529–556. Pergamon Press 1965. Printed in Great Britain

ON THE THEORY OF CONDUCTIVE HEAT TRANSFER IN FINITE REGIONS WITH BOUNDARY CONDITIONS OF THE SECOND KIND

NURETTİN Y. ÖLÇER
Ordnance Engineering Associates, Inc., Des Plaines, Illinois, U.S.A.

(*Received* 10 *August* 1964 *and in revised form* 26 *October* 1964)

Abstract—General expressions are derived for unsteady temperature distributions in finite regions of arbitrary geometry, under conditions of prescribed heat flux on all boundaries and with time-dependent heat sources and arbitrary initial conditions. The heat sources (or sinks) are distributed throughout the volume and can, as special cases, be surface, line or point sources. By introducing certain artificial additional heat source functions, corresponding pseudo-steady solutions are defined, by means of which the temperature fields are expressed in the form of uniformly convergent series solutions. The general method of solution is applied to a detailed study of a finite cylinder problem of very general nature, which has not been treated before.

The present work complements and supplements a previous paper in which assumption is made of the existence of steady-state solutions when prescribed volume and surface source functions are independent of time.

NOMENCLATURE

$A_i(s_i) \geqslant 0$, $B_i(s_i) \geqslant 0$,	boundary coefficient functions on S_i;
a,	radius of cylinder;
b,	half length of cylinder;
C_m,	coefficient defined in equation (11);
C_m^*,	coefficient defined in equation (7);
C_{kmn},	coefficient defined in equation (36);
D_{km},	coefficient defined in equation (65.b);
$F(P)$,	initial temperature distribution in R;
$F(r, \varphi, z)$,	initial temperature distribution in cylinder;
$f_i(s_i, t)$,	source function on S_i;
$f_1(r, \varphi, t)$,	heat flux per unit time on surface $z = -b$ of cylinder;
$f_2(r, \varphi, t)$,	heat flux per unit time on surface $z = b$ of cylinder;
$f_3(\varphi, z, t)$,	heat flux per unit time on surface $r = a$ of cylinder;
$G_{kn}(r, \rho)$,	Green's function defined in equation (52);
$I_k(x)$,	modified Bessel function of the first kind of order k and argument x;
i,	$1, 2, \ldots, q$;
$J_k(x)$	Bessel function of the first kind of order k and argument x;
j,	$0, 1, 2, \ldots, q$;
K,	thermal conductivity of R;
$K_k(x)$,	modified Bessel function of the second kind of order k and argument x;
k, m, n,	$0, 1, 2, \ldots, \infty$;
n_i,	outward normal of S_i;
P,	point in R;
$Q(P, t)$,	internal heat source function per unit time per unit volume of R;
$Q(r, \varphi, z, t)$	internal heat source function per unit time per unit volume of cylinder;

q,	number of co-ordinate surfaces of R;
R,	homogeneous, stationary region in P-space;
r,	radial space co-ordinate in cylindrical polar co-ordinates;
S,	boundary of R;
S_i,	i^{th} co-ordinate surface of R;
s_i,	point on S_i;
$T(P, t)$,	unsteady temperature distribution defined in equations (1), (2) and (3);
$T^*(P, t)$,	unsteady temperature distribution defined in equations (1), (2′) and (3), with $T(P, t)$ replaced by $T^*(P, t)$ in equations (1) and (3);
$T_{0j}(P, t)$,	pseudo-steady temperature distributions of order zero, defined in equations (19) and (20);
$T^*_{0j}(P, t)$,	pseudo-steady temperature distributions of order zero defined in equation (5);
$T_1(P, t)$,	unsteady temperature distribution defined in equation (24) or (25);
$T_2(P, t)$,	unsteady temperature distribution defined in equation (28);
$T_{av}(t)$,	average temperature of R at time t;
$T(r, \varphi, z, t)$,	unsteady temperature distribution defined in equations (29), (30), and (31);
$T_{00}(r, \varphi, z, t)$,	pseudo-steady temperature distribution defined in equations (39), (40), and (41), and given by equation (54);
$T_{01}(r, \varphi, z, t)$,	pseudo-steady temperature distribution defined in equations (56), (57) and (58), and given by equation (62) or (70);
$T_{02}(r, \varphi, z, t)$,	pseudo-steady temperature distribution defined in equations (73), (74) and (75), and given by equation (76) or (78);
$T_{03}(r, \varphi, z, t)$,	pseudo-steady temperature distribution defined in equations (81), (82) and (83), and given by equation (84);
$T_{00}(r, z, t)$,	axisymmetric pseudo-steady temperature distribution given by equation (55.a);
$T_{00}(r, \varphi, t)$,	two-dimensional pseudo-steady temperature distribution given by equation (55.b);
$T_{01}(r, z, t)$,	axisymmetric pseudo-steady temperature distribution given by equation (63.a) or (71.a);
$T_{02}(r, z, t)$,	axisymmetric pseudo-steady temperature distribution given by equation (77.a) or (79.a);
$T_{03}(r, z, t)$,	axisymmetric pseudo-steady temperature distribution given by equation (85.a);
$T_{03}(r, \varphi, t)$,	two-dimensional pseudo-steady temperature distribution given by equation (85.b) or (99);
$T(r, t)$,	one-dimensional unsteady temperature distribution given by equation (86);
$T(z, t)$,	one-dimensional unsteady temperature distribution given by equation (87) or (98);
$T_{00}(r, t)$,	one-dimensional pseudo-steady temperature distribution given by equation (55.c);
$T_{01}(z, t)$,	one-dimensional pseudo-steady temperature distribution given by equation (63.b) or (71.b);
$T_{02}(z, t)$,	one-dimensional pseudo-steady temperature distribution given by equation (77.b) or (79.b);
$T_{03}(r, t)$,	one-dimensional pseudo-steady temperature distribution given by equation (85.c);

t,	time variable;
V,	volume of R;
z,	axial co-ordinate in cylindrical polar co-ordinates;
δ_{ij}, δ_{0j}, δ_{k0}, δ_{n0}, δ_{m0},	Kronecker delta;
$\partial/\partial t$,	partial derivative with respect to time;
$\Theta_j(P, \tau, t)$,	unsteady temperature distributions defined by equation (26) or (27);
κ,	thermal diffusivity of R;
$\lambda_m \geqslant 0$,	eigenvalues defined by equation (10);
$\lambda_m^* > 0$,	eigenvalues defined by equation (6);
$\lambda_{kmn} \geqslant 0$,	eigenvalues defined by equation (32) and given by equation (34);
$\mu_{km} \geqslant 0$,	eigenvalues determined from equation (35);
τ,	parameter and variable of integration with respect to time;
$\phi_m(P)$,	eigenfunctions defined by equation (10);
$\phi_m^*(P)$,	eigenfunctions defined by equation (6);
$\phi_{kmn}(r, \varphi, z)$,	eigenfunctions defined by equation (32);
φ,	angular space co-ordinate in cylindrical polar co-ordinates;
$\Omega_j(t)$,	temperature functions defined by equation (16) and by equation (37) for $q = 3$;
∇,	gradient operator in P-space;
∇^2,	Laplace operator in P-space;
$\dot{(\ )}$	$[\partial(\ )/\partial t]$ or $[\partial(\ )/\partial\tau]$;
$\widetilde{\{\ \}}(r, k, z, t; \varphi')$,	finite cosine transform of $\{\ \}(r, \varphi, z, t)$ defined as $\int_0^{2\pi} \{\ \}(r, \varphi, z, t) \cos k(\varphi - \varphi')\, d\varphi$;
$\overline{\{\ \}}(r, \varphi, n, t)$,	finite cosine transform of $\{\ \}(r, \varphi, z, t)$ defined as $\int_{-b}^{b} \{\ \}(r, \varphi, z, t) \cos (n\pi/2) [1 + (z/b)]\, dz$;
$\hat{\{\ \}}(m, \varphi, z, t)$,	finite Hankel transform of $\{\ \}(r, \varphi, z, t)$ defined as $\int_0^a \{\ \}(r, \varphi, z, t) J_k(\mu_{km} r)\, r\, dr$.

INTRODUCTION

IN A RECENT PAPER [1] the author presented a general study of unsteady temperature distributions in finite regions of arbitrary geometry, under a wide variety of time-dependent boundary conditions and heat sources. The general solution was given in terms of a set of pseudo-steady temperature distributions and a uniformly convergent infinite series. It was remarked there that when all the boundaries of the finite region in question are simultaneously subjected to prescribed heat flux conditions, these pseudo-steady solutions do not exist (except in a trivial case of no practical significance) and therefore special attention must be devoted to the case of the boundary conditions of the second kind. It is the purpose of the present paper to present a study applicable to this important case of permanent interest to engineering science, and thus to complement and supplement the previous treatment given in [1].

STATEMENT OF THE PROBLEM

The unsteady temperature field in a stationary, homogeneous, isotropic region R, with thermal properties independent of temperature satisfies the heat-conduction equation

$$\nabla^2 T(P, t) + \frac{1}{K} Q(P, t) = \frac{1}{\kappa} \frac{\partial T(P, t)}{\partial t}, \quad P \text{ in } R, \quad t > 0. \tag{1}$$

Let the boundary surface S of R be composed of continuous co-ordinate surfaces S_i, q in number, in a conveniently chosen three-dimensional co-ordinate system. The boundary conditions of the second kind can be expressed as

$$K\frac{\partial T(P,t)}{\partial n_i}=f_i(s_i,t),\quad P \text{ on } S_i,\quad t>0, \tag{2}$$

and the initial condition is given by

$$T(P,t)=F(P),\quad P \text{ in } R,\quad t=0. \tag{3}$$

For ease of reference and comparison, the solution of the general problem treated in [1] where (2) is replaced by

$$A_i(s_i)\frac{\partial T^*(P,t)}{\partial n_i}+B_i(s_i)\,T^*(P,t)=f_i(s_i,t),\quad P \text{ on } S_i,\quad t>0, \tag{2'}$$

is also presented here as follows:

$$\left.\begin{aligned}T^*(P,t)=\sum_{j=0}^{q}T^*_{0j}(P,t)+\sum_{m=1}^{\infty}C^*_m\,\phi^*_m(P)\exp(-\kappa\lambda^{*2}_m t)\,\{\int_R\phi^*_m(P)\,F(P)\,\mathrm{d}V\\ -\sum_{j=0}^{q}[\int_R\phi^*_m(P)\,T^*_{0j}(P,0)\,\mathrm{d}V+\int_0^t\exp(\kappa\lambda^{*2}_m\tau)\int_R\phi^*_m(P)\,\dot{T}^*_{0j}(P,\tau)\,\mathrm{d}V\,\mathrm{d}\tau]\},\end{aligned}\right\} \tag{4.a}$$

where

$$\nabla^2 T^*_{0j}(P,t)+\frac{\delta_{0j}}{K}Q(P,t)=0,\quad P \text{ in } R, \tag{5.a}$$

$$A_i(s_i)\frac{\partial T^*_{0j}(P,t)}{\partial n_i}+B_i(s_i)\,T^*_{0j}(P,t)=\delta_{ij}f_i(s_i,t),\quad P \text{ on } S_i, \tag{5.b}$$

$$\nabla^2\phi^*_m(P)+\lambda^{*2}_m\,\phi^*_m(P)=0,\quad P \text{ in } R, \tag{6.a}$$

$$A_i(s_i)\frac{\partial\phi^*_m(P)}{\partial n_i}+B_i(s_i)\,\phi^*_m(P)=0,\quad P \text{ on } S_i, \tag{6.b}$$

and

$$\frac{1}{C^*_m}=\int_R\phi^{*2}_m(P)\,\mathrm{d}V. \tag{7}$$

An alternative expression for $T^*(P,t)$ is given by

$$\left.\begin{aligned}T^*(P,t)=\sum_{j=0}^{q}T^*_{0j}(P,t)+\sum_{m=1}^{\infty}C^*_m\,\phi^*_m(P)\exp(-\kappa\lambda^{*2}_m t)\left\{\int_R\phi^*_m(P)\,F(P)\,\mathrm{d}V\right.\\ -\frac{1}{\lambda^{*2}_m}\left[\frac{1}{K}\int_R\phi^*_m(P)\,Q(P,0)\,\mathrm{d}V+\sum_{i=1}^{q}\int_{S_i}\frac{\phi^*_m(s_i)}{A_i(s_i)}f_i(s_i,0)\,\mathrm{d}S_i\right]\\ \left.-\frac{1}{\lambda^{*2}_m}\int_0^t\exp(\kappa\lambda^{*2}_m\tau)\left[\frac{1}{K}\int_R\phi^*_m(P)\,\dot{Q}(P,\tau)\,\mathrm{d}V+\sum_{i=1}^{q}\int_{S_i}\frac{\phi^*_m(s_i)}{A_i(s_i)}f_i(s_i,\tau)\,\mathrm{d}S_i\right]\mathrm{d}\tau\right\}.\end{aligned}\right\} \tag{4.b}$$

The temperature distributions $T^*_{0j}(P,t)$ defined by the system (5) are called pseudo-steady solutions of order zero and satisfy the relations

$$\int_R\phi^*_m(P)\,T^*_{00}(P,t)\,\mathrm{d}V=\frac{1}{K\lambda^{*2}_m}\int_R\phi^*_m(P)\,Q(P,t)\,\mathrm{d}V, \tag{8.a}$$

$$\int_R \phi^*_m(P)\, T^*_{0j}(P, t)\, \mathrm{d}V = \frac{1}{\lambda^{*2}_m} \int_{S_j} \frac{\phi^*_m(s_j)}{A_j(s_j)} f_j(s_j, t)\, \mathrm{d}S_j, \quad j \neq 0. \tag{8.b}$$

Under the restriction that $B_i(s_i)$ is not to vanish for all i simultaneously, the expressions (4) constitute the solutions to the system of (1), (2′) and (3), and $T^*(P, t)$ satisfies the relation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_R \phi^*_m(P)\, T^*(P, t)\, \mathrm{d}V = \kappa \lambda^{*2}_m \int_R \phi^*_m(P) \left[\sum_{j=0}^{q} T^*_{0j}(P, t) - T^*(P, t)\right] \mathrm{d}V. \tag{9}$$

In the case where $B_i(s_i) \equiv 0$ for all i simultaneously, namely, the case of boundary conditions of the second kind, the system of (1), (2′) and (3) reduces to the system of (1), (2) and (3). It was noted in [1] that in this case the pseudo-steady solutions $T^*_{0j}(P, t)$ defined by the system (5) do not exist except when

$$\delta_{0j} \int_R Q(P, t)\, \mathrm{d}V + K\, \delta_{ij} \int_{S_i} \frac{f_i(s_i, t)}{A_i(s_i)}\, \mathrm{d}S_i = 0,$$

i.e. when the net rate of total heat transfer throughout the volume is zero. Therefore, for arbitrary source functions, $Q(P, t)$ and $f_i(s_i, t)$ the solution $T(P, t)$ to the system of (1), (2) and (3) cannot be obtained from the solution $T^*(P, t)$ to the system of (1), (2′) and (3) as expressed by (4).

SOLUTION OF THE PROBLEM

For the problem at hand the appropriate eigenvalue problem is the one defined by

$$\nabla^2 \phi_m(P) + \lambda^2_m\, \phi_m(P) = 0, \quad P \text{ in } R, \tag{10.a}$$

$$\frac{\partial \phi_m(P)}{\partial n_i} = 0, \quad P \text{ on } S_i, \tag{10.b}$$

and the normalizing coefficient C_m is given by

$$\frac{1}{C_m} = \int_R \phi^2_m(P)\, \mathrm{d}V. \tag{11}$$

It was shown in [1] that the eigenvalues λ_m and the eigenfunctions $\phi_m(P)$ defined by the system (10) satisfy the relation

$$\lambda^2_m = C_m \int_R [\nabla\, \phi_m(P)]^2\, \mathrm{d}V. \tag{12}$$

It follows from (12), that $\lambda_0 = 0$ is also an eigenvalue of (10) corresponding to the eigenfunction $\phi_0 = \text{constant} \neq 0$ $(m = 0)$.

In terms of the eigenfunctions $\phi_m(P)$ the following expansion can be written:

$$T(P, t) = \sum_{j=0}^{q} T_{0j}(P, t) + \sum_{m=0}^{\infty} C_m\, \phi_m(P) \int_R \phi_m(P)\, [T(P, t) - \sum_{j=0}^{q} T_{0j}(P, t)]\, \mathrm{d}V, \tag{13}$$

where $T_{0j}(P, t)$ are the pseudo-steady temperature distributions in the case of the boundary conditions of the second kind, and are to be defined presently. From the system of (1), (2), (3) and the system (10), it can be shown that for $\lambda_m = \lambda_0 = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_R T(P, t)\, \mathrm{d}V = \frac{\kappa}{K} \left[\int_R Q(P,t)\, \mathrm{d}V + \sum_{i=1}^{q} \int_{S_i} f_i(s_i, t)\, \mathrm{d}S_i\right], \tag{14.a}$$

the integration of which, along with the use of (3), gives

$$\int_R T(P,t)\,\mathrm{d}V = \int_R F(P)\,\mathrm{d}V + \frac{\kappa}{K}\int_0^t \left[\int_R Q(P,\tau)\,\mathrm{d}V + \sum_{i=1}^{q}\int_{S_i} f_i(s_i,\tau)\,\mathrm{d}S_i\right]\mathrm{d}\tau. \tag{14.b}$$

In view of (14.b), expression (13) is rewritten as

$$\left.\begin{aligned} T(P,t) = \frac{1}{V}\int_R F(P)\,\mathrm{d}V &+ \sum_{j=0}^{q}[\Omega_j(t) + T_{0j}(P,t)] \\ &+ \sum_{m=1}^{\infty} C_m\phi_m(P)\int_R \phi_m(P)\left[T(P,t) - \sum_{j=0}^{q} T_{0j}(P,t)\right]\mathrm{d}V, \end{aligned}\right\} \tag{15}$$

where

$$\Omega_j(t) = \frac{\kappa}{KV}\left[\delta_{0j}\int_0^t\int_R Q(P,\tau)\,\mathrm{d}V\,\mathrm{d}\tau + \delta_{ij}\int_0^t\int_{S_i} f_i(s_i,\tau)\,\mathrm{d}S_i\,\mathrm{d}\tau\right]. \tag{16}$$

From (14.b) and (16) it follows that

$$\left.\begin{aligned} \sum_{j=0}^{q}\Omega_j(t) &= \frac{1}{V}\int_R T(P,t)\,\mathrm{d}V - \frac{1}{V}\int_R F(P)\,\mathrm{d}V, \\ &= T_{\mathrm{av}}(t) - T_{\mathrm{av}}(0). \end{aligned}\right\} \tag{17}$$

Thus, the physical interpretation of $\sum_{j=0}^{q}\Omega_j(t)$ is that it represents the difference between the space-average temperature at time t and the initial average temperature of region R.

In order to determine the differential equations satisfied by the $T_{0j}(P,t)$ functions it is required that they satisfy

$$\int_R \phi_m(P)\,T_{0j}(P,t)\,\mathrm{d}V = \frac{1}{K\lambda_m^2}\left[\delta_{0j}\int_R \phi_m(P)\,Q(P,t)\,\mathrm{d}V + \delta_{ij}\int_{S_i}\phi_m(s_i)f_i(s_i,t)\,\mathrm{d}S_i\right], \quad \lambda_m \neq 0. \tag{18}$$

It should be noted that in writing (15) it has been tacitly required that

$$\int_R T_{0j}(P,t)\,\mathrm{d}V = 0. \tag{19}$$

Now the Laplacian ∇^2 is applied formally to both sides of (15) to yield, in view of (10.a), (18) and (9) as applied to $T(P,t)$ and $T_{0j}(P,t)$, the equation

$$\nabla^2 T(P,t) = \sum_{j=0}^{q}\nabla^2 T_{0j}(P,t) + \frac{1}{\kappa}\sum_{m=1}^{\infty} C_m\,\phi_m(P)\frac{\mathrm{d}}{\mathrm{d}t}\int_R \phi_m(P)\,T(P,t)\,\mathrm{d}V.$$

Using the expansion formula (13) with $T_{0j}(P,t)$ replaced by zero, and taking into account (14.a) and (16), the above result becomes

$$\nabla^2 T(P,t) = \sum_{j=0}^{q}\nabla^2 T_{0j}(P,t) + \frac{1}{\kappa}\frac{\partial T(P,t)}{\partial t} - \frac{1}{\kappa}\sum_{j=0}^{q}\frac{\mathrm{d}\Omega_j(t)}{\mathrm{d}t},$$

which, in view of (1), reduces to

$$\sum_{j=0}^{q}\left[\nabla^2 T_{0j}(P,t)+\frac{\delta_{0j}}{K}Q(P,t)-\frac{1}{\kappa}\frac{\mathrm{d}\Omega_j(t)}{\mathrm{d}t}\right]=0.$$

Thus the pseudo-steady temperature distributions $T_{0j}(P, t)$ satisfy the differential equations

$$\nabla^2 T_{0j}(P,t)+\frac{\delta_{0j}}{K}Q(P,t)=\frac{1}{\kappa}\frac{\mathrm{d}\Omega_j(t)}{\mathrm{d}t},\quad P \text{ in } R. \tag{20.a}$$

It is further required that they satisfy the boundary conditions

$$K\frac{\partial T_{0j}(P,t)}{\partial n_i}=\delta_{ij}f_i(s_i,t),\quad P \text{ on } S_i. \tag{20.b}$$

Using (10) and (20), it can be shown that the $T_{0j}(P, t)$ functions satisfy (18). By utilizing the Gauss Divergence Theorem it can also be shown from (20) that (19) is satisfied.

The pseudo-steady solutions $T_{0j}(P, t)$ comprise essentially a set of fictitious steady-state temperature distributions in which t is regarded as a parameter. They are to be determined from the system (20), subject to the conditions of (19). These should be compared with (5), the defining system for $T^*_{0j}(P, t)$. In view of (18), the series expression for $T_{0j}(P, t)$ is

$$T_{0j}(P,t)=\frac{1}{K}\sum_{m=1}^{\infty}\frac{C_m}{\lambda_m^2}\phi_m(P)\left[\delta_{0j}\int_R \phi_m(P)\,Q(P,t)\,\mathrm{d}V+\delta_{ij}\int_{S_i}\phi_m(s_i)f_i(s_i,t)\,\mathrm{d}S_i\right]. \tag{21}$$

It should be noted that although $\lambda_0 = 0$ is also an eigenvalue of (10), expression (21) does not include the term corresponding to $m = 0$, since $T_{0j}(P, t)$ satisfies (19). If the $T_{0j}(P, t)$ functions can be determined directly from (20) and (19), the above eigenfunction expansions for $T_{0j}(P, t)$ constitute a set of summation formulas, and the similar other concluding remarks of [1] apply *mutatis mutandis* to the expression (21).

Next, (9) is integrated with respect to t, after replacing $\phi^*_m(P)$, $T^*(P, t)$ and $T^*_{0j}(P, t)$ by $\phi_m(P)$ $T(P, t)$ and $T_{0j}(P, t)$, respectively, and the result introduced into (15) to give the following expression for the temperature distribution $T(P, t)$:

$$\left.\begin{aligned}T(P,t)&=\frac{1}{V}\int_R F(P)\,\mathrm{d}V+\sum_{j=0}^{q}[\Omega_j(t)+T_{0j}(P,t)]\\&\quad+\sum_{m=1}^{\infty}C_m\,\phi_m(P)\exp(-\kappa\lambda_m^2 t)\left\{\int_R \phi_m(P)\left[F(P)-\sum_{j=0}^{q}T_{0j}(P,0)\right]\mathrm{d}V\right.\\&\qquad\left.-\sum_{j=0}^{q}\int_0^t \exp(\kappa\lambda_m^2\tau)\int_R \phi_m(P)\,\dot{T}_{0j}(P,\tau)\,\mathrm{d}V\,\mathrm{d}\tau\right\}.\end{aligned}\right\} \tag{22.a}$$

An alternate expression obtained by substituting (16) and (18) in (22a) is as follows:

$$\left.\begin{aligned}T(P,t)&=\frac{1}{V}\int_R F(P)\,\mathrm{d}V+\frac{\kappa}{KV}\int_0^t\left[\int_R Q(P,\tau)\,\mathrm{d}V+\sum_{i=1}^{q}\int_{S_i}f_i(s_i,\tau)\,\mathrm{d}S_i\right]\mathrm{d}\tau\\&\quad+\sum_{j=0}^{q}T_{0j}(P,t)+\sum_{m=1}^{\infty}C_m\,\phi_m(P)\exp(-\kappa\lambda_m^2 t)\left\{\int_R \phi_m(P)\,F(P)\,\mathrm{d}V\right.\end{aligned}\right\} \tag{22.b}$$

$$\left.\begin{aligned} & - \frac{1}{K\lambda_m^2}\left[\int_R \phi_m(P)\, Q(P,0)\, \mathrm{d}V + \sum_{i=1}^{q} \int_{S_i} \phi_m(s_i)\, f_i(s_i,0)\, \mathrm{d}S_i\right] \\ & - \int_0^t \frac{\exp(\kappa\lambda_m^2\tau)}{K\lambda_m^2}\left[\int_R \phi_m(P)\, \dot{Q}(P,\tau)\, \mathrm{d}V + \sum_{i=1}^{q}\int_{S_i} \phi_m(s_i)\, \dot{f}_i(s_i,\tau)\, \mathrm{d}S_i\right] \mathrm{d}\tau\Big\}. \end{aligned}\right\} \quad (22.\text{b})$$

In this second solution form for $T(P, t)$, the source functions $Q(P, t)$ and $f_i(s_i, t)$ appear explicitly. Direct substitution shows that the solutions (22) satisfy the differential equation (1), the boundary conditions (2) and the initial condition (3).

REMARKS

In the derivation of (22) it is tacitly assumed that the source functions $F(P)$, $Q(P, t)$ and $f_i(s_i, t)$ are integrable with respect to their independent variables. To guarantee the uniform convergence of the infinite series in (22) it is further assumed that $F(P)$, $Q(P, t)$ and $f_i(s_i, t)$ possess continuous first and second order partial derivatives with respect to space variables, and that the time-dependent source functions $Q(P, t)$ and $f_i(s_i, t)$ possess continuous first order partial derivatives with respect to t.

The difference between the solutions to the system of (1), (2) and (3) as expressed by (22), and those to the system of (1), (2′), and (3) as expressed by (4) can be seen by a comparison of the two. The difference is not only due to the explicit presence of the additional $\Omega_j(t)$ and $(1/V)\int_R F(P)\,\mathrm{d}V$ terms in (22) but also due to the implicit effect of the $\Omega_j(t)$ terms on the solutions (22), since the latter also appear, as fictitious volume sources, in the differential equations (20.a) for the $T_{0j}(P, t)$ functions. In addition, the zeroth order pseudo-steady solutions $T_{0j}(P, t)$ are subject to the conditions expressed by (19) without which they are indeterminate. As shown by (17), the additional terms of

$$\frac{1}{V}\int_R F(P)\,\mathrm{d}V + \sum_{j=0}^{q} \Omega_j(t)$$

appearing in (22) represent the space-average temperature throughout region R and constitute a characteristic feature of the solution of heat conduction equation in a finite region the entire surface of which is subjected to boundary conditions of the second kind. This is of course obvious from the physical nature of the problem.

As in the case of [1], the general solution (22) can be expressed in a more compact form as

$$T(P,t) = T_1(P,t) + \sum_{j=0}^{q}\left\{\Omega_j(t) + T_{0j}(P,t) - \int_0^t \left[\frac{\partial\Theta_j(P,\tau',t-\tau)}{\partial\tau'}\right]_{\tau'=\tau} \mathrm{d}\tau\right\}. \quad (23)$$

In this expression $T_1(P, t)$ is the solution of

$$\left.\begin{aligned} & \nabla^2 T_1(P,t) = \frac{1}{\kappa}\frac{\partial T_1(P,t)}{\partial t}, \quad P \text{ in } R, \quad t>0, \\ & \text{with} \\ & K\frac{\partial T_1(P,t)}{\partial n_i} = 0, \quad P \text{ on } S_i, \quad t>0, \\ & \text{and} \\ & T_1(P,t) = F(P) - \sum_{j=0}^{q} T_{0j}(P,0), \quad P \text{ in } R, \quad t=0, \end{aligned}\right\} \quad (24)$$

and is given by

$$T_1(P,t) = \frac{1}{V}\int_R F(P)\,\mathrm{d}V + \sum_{m=1}^{\infty} C_m\,\phi_m(P)\exp(-\kappa\lambda_m^2 t)\int_R \phi_m(P)\left[F(P) - \sum_{j=0}^{q} T_{0j}(P,0)\right]\mathrm{d}V, \tag{25.a}$$

or

$$\left.\begin{aligned} T_1(P,t) = \frac{1}{V}\int_R F(P)\,\mathrm{d}V + \sum_{m=1}^{\infty} C_m\,\phi_m(P)\exp(-\kappa\lambda_m^2 t)\left\{\int_R \phi_m(P)\,F(P)\,\mathrm{d}V\right. \\ \left. - \frac{1}{K\lambda_m^2}\left[\int_R \phi_m(P)\;Q(P,0)\,\mathrm{d}V + \sum_{i=1}^{q}\int_{S_i} \phi_m(s_i)\,f_i(s_i,0)\,\mathrm{d}S_i\right]\right\}. \end{aligned}\right\} \tag{25.b}$$

Similarly, $\Theta_j(P,\tau,t)$ are the solutions of

$$\left.\begin{aligned} &\nabla^2\Theta_j(P,\tau,t) = \frac{1}{\kappa}\frac{\partial\Theta_j(P,\tau,t)}{\partial t}, \quad P \text{ in } R, \quad t>0, \\ &\text{with} \\ &K\frac{\partial\Theta_j(P,\tau,t)}{\partial n_i} = 0, \quad P \text{ on } S_i, \quad t>0, \\ &\text{and} \\ &\Theta_j(P,\tau,t) = T_{0j}(P,\tau), \quad P \text{ in } R, \quad t=0, \end{aligned}\right\} \tag{26}$$

and are given by

$$\Theta_j(P,\tau,t) = \sum_{m=1}^{\infty} C_m\,\phi_m(P)\exp(-\kappa\lambda_m^2 t)\int_R \phi_m(P)T_{0j}(P,\tau)\,\mathrm{d}V, \tag{27.a}$$

or

$$\Theta_j(P,\tau,t) = \sum_{m=1}^{\infty} C_m\,\phi_m(P)\frac{\exp(-\kappa\lambda_m^2 t)}{K\lambda_m^2}\left[\delta_{0j}\int_R \phi_m(P)\;Q(P,\tau)\,\mathrm{d}V + \delta_{ij}\int_{S_i}\phi_m(s_i)\,f_i(s_i,\tau)\,\mathrm{d}S\right]. \tag{27.b}$$

It should be noted that use has been made of (19) in writing (25) and (27). Thus, $[T_1(P,t) - T_{\mathrm{av}}(0)]$ and $\Theta_j(P,\tau,t)$ constitute entirely transient solutions composed of exponentially decaying terms, and satisfy homogeneous differential equations and homogeneous boundary conditions. It follows from (23) and (25) that the quasi-steady temperature distribution is given by the asymptotic behavior of

$$T_2(P,t) = \frac{1}{V}\int_R F(P)\,\mathrm{d}V + \sum_{j=0}^{q}\left\{\Omega_j(t) + T_{0j}(P,t) - \int_0^t \left[\frac{\partial\Theta_j(P,\tau',t-\tau)}{\partial\tau'}\right]_{\tau'=\tau}\mathrm{d}\tau\right\} \tag{28}$$

for large values of time t.

APPLICATION

By way of illustration of an application of the foregoing results, a general problem that has not been solved before will now be considered. The problem is the determination of the unsteady temperature distribution in a right circular solid cylinder of finite length, with its entire surface subjected to boundary conditions of the second kind. Using cylindrical polar co-ordinates, r, φ, z, and choosing the z-co-ordinate along the geometrical axis of the cylinder, the region R is defined by

$$0 \leqslant r \leqslant a; \qquad 0 \leqslant \varphi \leqslant 2\pi; \qquad |z| \leqslant b.$$

Evidently, $q = 3$. Let $i = 1, 2, 3$ refer, respectively, to surfaces $z = -b$, $z = b$ and $r = a$ of the cylinder. Then, (1), (2) and (3) yield respectively,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right) T(r, \varphi, z, t) + \frac{1}{K} Q(r, \varphi, z, t) = \frac{1}{\kappa}\frac{\partial T(r, \varphi, z, t)}{\partial t},$$

$$(0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| < b, \quad t > 0), \tag{29}$$

$$-K\frac{\partial T(r, \varphi, z, t)}{\partial z} = f_1(r, \varphi, t), \quad (0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad z = -b, \quad t > 0), \tag{30.a}$$

$$K\frac{\partial T(r, \varphi, z, t)}{\partial z} = f_2(r, \varphi, t), \quad (0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad z = b, \quad t > 0), \tag{30.b}$$

$$K\frac{\partial T(r, \varphi, z, t)}{\partial r} = f_3(\varphi, z, t), \quad (r = a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| < b, \quad t > 0), \tag{30.c}$$

$$T(r, \varphi, z, t) = F(r, \varphi, z), \quad (0 \leqslant r \leqslant a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| \leqslant b, \quad t = 0). \tag{31}$$

The eigenvalue problem corresponding to (10) is expressed by

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} + \lambda_{kmn}^2\right) \phi_{kmn}(r, \varphi, z) = 0, \quad (0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| < b), \tag{32.a}$$

$$\left.\begin{aligned} -\frac{\partial \phi_{kmn}(r, \varphi, z)}{\partial z} &= 0, \quad (0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad z = -b), \\ \frac{\partial \phi_{kmn}(r, \varphi, z)}{\partial z} &= 0, \quad (0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad z = b), \\ \frac{\partial \phi_{kmn}(r, \varphi, z)}{\partial r} &= 0, \quad (r = a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| < b). \end{aligned}\right\} \tag{32.b}$$

The eigenfunctions of (32) well-behaved at $r = 0$ are obtained in terms of two triple index sets as

$$\phi_{kmn}(r, \varphi, z) = \left\{\begin{aligned} &J_k(\mu_{km} r)\cos k\varphi \cos \frac{n\pi}{2}\left(1 + \frac{z}{b}\right), \\ &J_k(\mu_{km} r)\sin k\varphi \cos \frac{n\pi}{2}\left(1 + \frac{z}{b}\right) \end{aligned}\right\} (k, m, n = 0, 1, 2, \ldots), \tag{33}$$

and the eigenvalues λ_{kmn} are given by

$$\lambda_{kmn}^2 = \mu_{km}^2 + \left(\frac{n\pi}{2b}\right)^2, \tag{34}$$

where $a\mu_{km} \geqslant 0$ is the m^{th} root of

$$(\mu_{km} a) J_k'(\mu_{km} a) = 0, \tag{35.a}$$

or of

$$(\mu_{km} a) J_{k+1}(\mu_{km} a) = k J_k(\mu_{km} a), \tag{35.b}$$

and the prime in (35.a) denotes differentiation with respect to the argument. It is to be noted that for

$k \neq 0$, $\mu_{km} = \mu_{k0} = 0$ is not an eigenvalue of (32). Zeros of $J'_k(x)$ for orders $0 \leqslant k \leqslant 20$ are contained in [2], and for orders $21 \leqslant k \leqslant 51$ and $0 \leqslant x \leqslant 100$ are tabulated in [3].

From (11) and (33),

$$\left.\begin{aligned} \frac{1}{C_{kmn}} &= \int_0^a \int_0^{2\pi} \int_{-b}^{b} J_k^2(\mu_{km} r) \begin{Bmatrix} \cos^2 k\varphi \\ \sin^2 k\varphi \end{Bmatrix} \cos^2 \frac{n\pi}{2}\left(1+\frac{z}{b}\right) r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z \\ &= \frac{\pi}{2} a^2 b (1+\delta_{k0})(1+\delta_{n0}) \left[1 - \left(\frac{k}{\mu_{km} a}\right)^2\right] J_k^2(\mu_{km} a). \end{aligned}\right\} \tag{36}$$

Since $V = 2\pi a^2 b$, (16) yields

$$\Omega_0(t) = \frac{1}{2\pi a^2 b} \cdot \frac{\kappa}{K} \int_0^t \int_0^a \int_0^{2\pi} \int_{-b}^{b} Q(r, \varphi, z, \tau)\, r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z\,\mathrm{d}\tau, \tag{37.a}$$

$$\Omega_1(t) = \frac{1}{2\pi a^2 b} \cdot \frac{\kappa}{K} \int_0^t \int_0^a \int_0^{2\pi} f_1(r, \varphi, \tau)\, r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\tau, \tag{37.b}$$

$$\Omega_2(t) = \frac{1}{2\pi a^2 b} \cdot \frac{\kappa}{K} \int_0^t \int_0^a \int_0^{2\pi} f_2(r, \varphi, \tau)\, r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\tau, \tag{37.c}$$

$$\Omega_3(t) = \frac{1}{2\pi a b} \cdot \frac{\kappa}{K} \int_0^t \int_0^{2\pi} \int_{-b}^{b} f_3(\varphi, z, \tau)\,\mathrm{d}\varphi\,\mathrm{d}z\,\mathrm{d}\tau. \tag{37.d}$$

From (22.a), (33) and (36), it follows that

$$\left.\begin{aligned} T(r, \varphi, z, t) &= \frac{1}{2\pi a^2 b} \int_0^a \int_0^{2\pi} \int_{-b}^{b} F(r, \varphi, z)\, r\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}z + \sum_{j=0}^{3} [\Omega_j(t) + T_{0j}(r, \varphi, z, t)] \\ &+ \frac{2}{\pi a^2 b} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\substack{n=0 \\ (k=m=n\neq 0)}}^{\infty} \frac{J_k(\mu_{km} r) \cos(n\pi/2)(1+z/b) \exp(-\kappa \lambda_{kmn}^2 t)}{(1+\delta_{k0})(1+\delta_{n0})[1-(k/\mu_{km} a)^2] J_k^2(\mu_{km} a)} \\ &\int_0^{2\pi} \left\{ \int_0^a \int_{-b}^{b} J_k(\mu_{km} r) \cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right) \left[F(r, \varphi', z) - \sum_{j=0}^{3} T_{0j}(r, \varphi', z, 0)\right] r\,\mathrm{d}r\,\mathrm{d}z \right. \\ &\left. - \sum_{j=0}^{3} \int_0^t \exp(\kappa \lambda_{kmn}^2 \tau) \int_0^a \int_{-b}^{b} J_k(\mu_{km} r) \cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right) \dot{T}_{0j}(r, \varphi', z, \tau)\, r\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\tau \right\} \\ &\cos k(\varphi - \varphi')\,\mathrm{d}\varphi'. \end{aligned}\right\} \tag{38.a}$$

Similarly, from (22.b), (33) and (36) it follows that

$$
\left.
\begin{aligned}
&T(r, \varphi, z, t) = \frac{1}{2\pi a^2 b} \int_0^a \int_0^{2\pi} \int_{-b}^{b} F(r, \varphi, z)\, r\, \mathrm{d}r\, \mathrm{d}\varphi\, \mathrm{d}z + \sum_{j=0}^{3} [\Omega_j(t) + T_{0j}(r, \varphi, z, t)] \\
&+ \frac{2}{\pi a^2 b} \sum_{\substack{k=0 \\ (k=m=n\neq 0)}}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{J_k(\mu_{km} r) \cos(n\pi/2)(1 + z/b) \exp(-\kappa \lambda_{kmn}^2 t)}{(1+\delta_{k0})(1+\delta_{n0})[1-(k/\mu_{km}a)^2] J_k^2(\mu_{km}a)} \\
&\int_0^{2\pi} \left\{ \int_0^a \int_{-b}^{b} J_k(\mu_{km} r) \cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right)\left[F(r, \varphi', z) - \frac{Q(r, \varphi' z, 0)}{K\lambda_{kmn}^2}\right] r\, \mathrm{d}r\, \mathrm{d}z \right. \\
&- \frac{1}{K\lambda_{kmn}^2} \int_0^a J_k(\mu_{km} r)\, [f_1(r, \varphi', 0) + (-1)^n f_2(r, \varphi', 0)]\, r\, \mathrm{d}r - \frac{a J_k(\mu_{km} a)}{K\lambda_{kmn}^2} \\
&\int_{-b}^{b} \cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right) f_3(\varphi', z, 0)\, \mathrm{d}z - \frac{1}{K\lambda_{kmn}^2} \int_0^t \exp(\kappa\lambda_{kmn}^2 \tau) \\
&\left[\int_0^a \int_{-b}^{b} J_k(\mu_{km} r) \cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right) \dot{Q}(r, \varphi', z, \tau)\, r\, \mathrm{d}r\, \mathrm{d}z + \int_0^a J_k(\mu_{km} r) \right. \\
&[\dot{f}_1(r, \varphi', \tau) + (-1)^n \dot{f}_2(r, \varphi', \tau)]\, r\, \mathrm{d}r + a J_k(\mu_{km} a) \\
&\left. \left. \int_{-b}^{b} \cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right) \dot{f}_3(r, \varphi', \tau)\, \mathrm{d}z \right] \mathrm{d}\tau \right\} \cos k(\varphi - \varphi')\, \mathrm{d}\varphi'.
\end{aligned}
\right\} \quad (38.b)
$$

Thus, once the pseudo-steady temperatures $T_{0j}(r, \varphi, z, t)$ have been determined, the expressions (38) together with (37) give the unsteady temperature field in the cylinder. The $T_{0j}(r, \varphi, z, t)$ functions are determined by the repeated application of one-dimensional finite integral transforms.

Determination of $T_{00}(r, \varphi, z, t)$

From (19) and (20), with $j = 0$, the differential equation and conditions defining $T_{00}(r, \varphi, z, t)$ are

$$
\left.
\begin{aligned}
\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right) T_{00}(r, \varphi, z, t) + \frac{1}{K} Q(r, \varphi, z, t) = \frac{1}{\kappa}\frac{\mathrm{d}\Omega_0(t)}{\mathrm{d}t}, \\
(0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| < b),
\end{aligned}
\right\} \quad (39)
$$

$$- K\frac{\partial T_{00}(r, \varphi, z, t)}{\partial z} = 0, \quad (0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad z = -b), \qquad (40.a)$$

$$K\frac{T_{00}(r, \varphi, z, t)}{\partial z} = 0, \quad (0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad z = b), \qquad (40.b)$$

$$K\frac{\partial T_{00}(r, \varphi, z, t)}{\partial r} = 0, \quad (r = a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| < b), \qquad (40.c)$$

$$\int_0^a \int_0^{2\pi} \int_{-b}^{b} T_{00}(r, \varphi, z, t)\, r\, \mathrm{d}r\, \mathrm{d}\varphi\, \mathrm{d}z = 0. \qquad (41)$$

In order to solve $T_{00}(r, \varphi, z, t)$ from the system of (39), (40) and (41), first a finite cosine transform is defined as

$$\tilde{T}_{00}(r, k, z, t; \varphi') = \int_0^{2\pi} T_{00}(r, \varphi, z, t) \cos k(\varphi - \varphi')\, d\varphi. \tag{42}$$

The inverse transform of (42) is given by

$$T_{00}(r, \varphi, z, t) = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{\tilde{T}_{00}(r, k, z, t; \varphi)}{(1 + \delta_{k0})}. \tag{43}$$

The equations (39) and (40) are now transformed by (42) to give

$$\left.\begin{aligned} &\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{k^2}{r^2} + \frac{\partial^2}{\partial z^2}\right) \tilde{T}_{00}(r, k, z, t; \varphi') + \frac{1}{K}\tilde{Q}(r, k, z, t; \varphi') = \frac{2\pi}{\kappa}\,\delta_{k0}\,\frac{d\Omega_0(t)}{dt}, \\ &\qquad (0 \leqslant r < a, \quad |z| < b), \end{aligned}\right\} \tag{44}$$

$$-K\frac{\partial \tilde{T}_{00}(r, k, z, t; \varphi')}{\partial z} = 0, \quad (0 \leqslant r < a, \quad z = -b), \tag{45.a}$$

$$K\frac{\partial \tilde{T}_{00}(r, k, z, t; \varphi')}{\partial z} = 0, \quad (0 \leqslant r < a, \quad z = b), \tag{45.b}$$

$$K\frac{\partial \tilde{T}_{00}(r, k, z, t; \varphi')}{\partial r} = 0, \quad (r = a, \quad |z| < b). \tag{45.c}$$

In order to determine $\tilde{T}_{00}(r, k, z, t; \varphi')$, another finite cosine transform is defined as

$$\bar{\tilde{T}}_{00}(r, k, n, t;\ \varphi') = \int_{-b}^{b} \tilde{T}_{00}(r, k, z, t; \varphi') \cos\frac{n\pi}{2}\left(1 + \frac{z}{b}\right) dz, \tag{46}$$

the inversion formula for which can be expressed by

$$\tilde{T}_{00}(r, k, z, t;\ \varphi') = \frac{1}{b}\sum_{n=0}^{\infty} \frac{\bar{\tilde{T}}_{00}(r, k, n, t;\ \varphi')}{(1 + \delta_{n0})} \cos\frac{n\pi}{2}\left(1 + \frac{z}{b}\right). \tag{47}$$

The application of (46) to the system of (44) and (45), along with the use of the Gauss Divergence Theorem in the z-co-ordinate, results in

$$\left.\begin{aligned} &\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\frac{k^2}{r^2} + \frac{n^2\pi^2}{4b^2}\right)\right] \bar{\tilde{T}}_{00}(r, k, n, t;\ \varphi') + \frac{1}{K}\bar{\tilde{Q}}(r, k, n, t;\ \varphi') \\ &\qquad = \frac{4\pi b}{\kappa}\,\delta_{k0}\cdot\delta_{n0}\,\frac{d\Omega_0(t)}{dt}, \\ &\qquad (0 \leqslant r < a), \end{aligned}\right\} \tag{48}$$

$$K\frac{\partial \bar{\tilde{T}}_{00}(r, k, n, t;\ \varphi')}{\partial r} = 0, \quad (r = a) \tag{49}$$

In terms of transform notation, (41) can be rewritten as

$$\int_0^a \bar{\bar{T}}_{00}(r, 0, 0, t;\ \varphi')\, r\, \mathrm{d}r = 0. \tag{50}$$

The solution to the system of (48) and (49), well-behaved at $r = 0$, is

$$\left.\begin{aligned} \bar{\bar{T}}_{00}(r, k, n, t;\ \varphi') = \frac{1}{K}\left[\int_0^r G_{kn}(\rho, r)\,\bar{\bar{Q}}(\rho, k, n, t;\ \varphi')\,\rho\,\mathrm{d}\rho + \int_r^a G_{kn}(r, \rho)\,\bar{\bar{Q}}(\rho, k, n, t;\ \varphi')\,\rho\,\mathrm{d}\rho\right],\\ (k = n \neq 0), \end{aligned}\right\} \tag{51.a}$$

where

$$G_{kn}(r, \rho) = \frac{I_k(n\pi r/2b)}{I_k'(n\pi a/2b)}\left[K_k\left(\frac{n\pi\rho}{2b}\right) I_k'\left(\frac{n\pi a}{2b}\right) - I_k\left(\frac{n\pi\rho}{2b}\right) K_k'\left(\frac{n\pi a}{2b}\right)\right], \tag{52.a}$$

in which primes indicate differentiation with respect to the arguments and

$$\lim_{n\to 0}\,[G_{kn}(r, \rho)] = \frac{1}{2k}\left(\frac{r}{a}\right)^k\left[\left(\frac{\rho}{a}\right)^k + \left(\frac{a}{\rho}\right)^k\right], \quad (k \neq 0). \tag{52.b}$$

For $k = n = 0$ the solution to the system of (48) and (49) well-behaved at $r = 0$ is determined by use of (50) and is given by

$$\bar{\bar{T}}_{00}(r, 0, 0, t; \varphi') = \frac{1}{2a^2}\int_0^a Q^*(\rho, t)\,\rho^3\,\mathrm{d}\rho + \int_0^r Q^*(\rho, t)\ln\left(\frac{a}{r}\right)\rho\,\mathrm{d}\rho + \int_r^a Q^*(\rho, t)\ln\left(\frac{a}{\rho}\right)\rho\,\mathrm{d}\rho,$$

where

$$Q^*(r, t) = \frac{1}{K}\bar{\bar{Q}}(r, 0, 0, t; \varphi') - \frac{4\pi b}{\kappa}\frac{\mathrm{d}\Omega_0(t)}{\mathrm{d}t}.$$

The above expression for $\bar{\bar{T}}_{00}(r, 0, 0, t; \varphi')$ can be rewritten as

$$\left.\begin{aligned} \bar{\bar{T}}_{00}(r, 0, 0, t; \varphi') = \frac{1}{K}\int_0^{2\pi}\int_{-b}^{b}\left\{\int_0^a\left(\frac{\rho^2 + r^2}{2a^2} - \frac{3}{4}\right) Q(\rho, \varphi, z, t)\,\rho\,\mathrm{d}\rho + \int_0^r \ln\left(\frac{a}{r}\right) Q(\rho, \varphi, z, t)\,\rho\,\mathrm{d}\rho\right.\\ \left. + \int_r^a \ln\left(\frac{a}{\rho}\right) Q(\rho, \varphi, z, t)\,\rho\,\mathrm{d}\rho\right\}\mathrm{d}\varphi\,\mathrm{d}z. \end{aligned}\right\} \tag{51.b}$$

If $Q(r, \varphi, z, t)$ is independent of r, (51.b) gives $\bar{\bar{T}}_{00}(r, 0, 0, t; \varphi') = 0$, as expected. This serves as an independent check on (51.b). The combination of (43) and (47) yields for $T_{00}(r, \varphi, z, t)$ the expression

$$T_{00}(r, \varphi, z, t) = \frac{1}{\pi b}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\bar{\bar{T}}_{00}(r, k, n, t; \varphi)}{(1 + \delta_{k0})(1 + \delta_{n0})}\cos\frac{n\pi}{2}\left(1 + \frac{z}{b}\right). \tag{53}$$

Introducing (51) into (53),

$$
\left.
\begin{aligned}
T_{00}(r,\varphi,z,t) &= \frac{1}{4\pi bK}\int_0^{2\pi}\int_{-b}^{b}\left\{\int_0^a\left(\frac{\rho^2+r^2}{2a^2}-\frac{3}{4}\right)Q(\rho,\varphi',z,t)\,\rho\,\mathrm{d}\rho+\int_0^r\ln\left(\frac{a}{r}\right)Q(\rho,\varphi',z,t)\,\rho\,\mathrm{d}\rho\right.\\
&\quad\left.+\int_r^a\ln\left(\frac{a}{\rho}\right)Q(\rho,\varphi',z,t)\,\rho\,\mathrm{d}\rho\right\}\mathrm{d}\varphi\,\mathrm{d}z+\frac{1}{4\pi bK}\sum_{k=1}^{\infty}\int_0^{2\pi}\int_{-b}^{b}\left\{\left[\left(\frac{r}{a}\right)^k\right.\right.\\
&\quad\left.+\left(\frac{a}{r}\right)^k\right]\int_0^r\left(\frac{\rho}{a}\right)^k Q(\rho,\varphi',z,t)\rho\,\mathrm{d}\rho+\left(\frac{r}{a}\right)^k\int_r^a\left[\left(\frac{\rho}{a}\right)^k+\left(\frac{a}{\rho}\right)^k\right]Q(\rho,\varphi',z,t)\,\rho\,\mathrm{d}\rho\right\}\\
&\quad\frac{\cos k(\varphi-\varphi')}{k}\,\mathrm{d}\varphi'\,\mathrm{d}z+\frac{1}{\pi bK}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{\cos(n\pi/2)(1+z/b)}{(1+\delta_{k0})}\int_0^{2\pi}\int_{-b}^{b}\left\{\int_0^r G_{kn}(\rho,r)\,Q(\rho,\varphi',z,t)\,\rho\,\mathrm{d}\rho\right.\\
&\quad\left.+\int_r^a G_{kn}(r,\rho)\,Q(\rho,\varphi',z,t)\,\rho\,\mathrm{d}\rho\right\}\cos k(\varphi-\varphi')\cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right)\mathrm{d}\varphi'\,\mathrm{d}z.
\end{aligned}
\right\}\quad(54)
$$

It is noted here that in determining $\bar{T}_{00}(r, k, z, t; \varphi')$ from (44) and (45), a finite Hankel transform can be applied to (44) and (45) instead of the finite cosine transform of (46), and thus an alternate expression can be obtained for $T_{00}(r, \varphi, z, t)$. This procedure is illustrated in the determination of $T_{01}(r, \varphi, z, t)$ and $T_{02}(r, \varphi, z, t)$, and the corresponding expressions are given by (70) and (78), respectively.

From (54), the two-dimensional cases of pseudo-steady temperature distributions can be readily written down. Thus, if $Q(r, \varphi, z, t)$ is independent of φ, the problem becomes one of axial symmetry and only the terms corresponding to $k = 0$ contribute to the solution expressed by (54). Hence,

$$
\left.
\begin{aligned}
T_{00}(r,z,t) &= \frac{1}{2bK}\int_{-b}^{b}\left\{\int_0^a\left(\frac{\rho^2+r^2}{2a^2}-\frac{3}{4}\right)Q(\rho,z,t)\,\rho\,\mathrm{d}\rho+\int_0^r\ln\left(\frac{a}{r}\right)Q(\rho,z,t)\,\rho\,\mathrm{d}\rho\right.\\
&\quad\left.+\int_r^a\ln\left(\frac{a}{\rho}\right)Q(\rho,z,t)\,\rho\,\mathrm{d}\rho\right\}\mathrm{d}z+\frac{1}{bK}\sum_{n=1}^{\infty}\cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right)\int_{-b}^{b}\left\{\int_0^r G_{0n}(\rho,r)Q\rho,z,t)\,\rho\,\mathrm{d}\rho\right.\\
&\quad\left.+\int_r^a G_{0n}(r,\rho)\,Q(\rho,z,t)\,\rho\,\mathrm{d}\rho\right\}\cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right)\mathrm{d}z.
\end{aligned}
\right\}\quad(55.\mathrm{a})
$$

On the other hand, if $Q(r, \varphi, z, t)$ is independent of z, there is no axial conduction and (54) reduces to

$$
\left.
\begin{aligned}
T_{00}(r,\varphi,t) &= \frac{1}{2\pi K}\int_0^{2\pi}\left\{\int_0^a\left(\frac{\rho^2+r^2}{2a^2}-\frac{3}{4}\right)Q(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho+\int_0^r\ln\left(\frac{a}{r}\right)Q(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho\right.\\
&\quad\left.+\int_r^a\ln\left(\frac{a}{\rho}\right)Q(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho\right\}\mathrm{d}\varphi'+\frac{1}{2\pi K}\sum_{k=1}^{\infty}\int_0^{2\pi}\left\{\left[\left(\frac{a}{r}\right)^k+\left(\frac{r}{a}\right)^k\right]\int_0^r\left(\frac{\rho}{a}\right)^k Q(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho\right.\\
&\quad\left.+\left(\frac{r}{a}\right)^k\int_r^a\left[\left(\frac{\rho}{a}\right)^k+\left(\frac{a}{\rho}\right)^k\right]Q(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho\right\}\frac{\cos k(\varphi-\varphi')}{k}\,\mathrm{d}\varphi'.
\end{aligned}
\right\}\quad(55.\mathrm{b})
$$

If $Q(r, \varphi, z, t)$ is independent of both φ and z, (54) simplifies to

$$T_{00}(r, t) = \frac{1}{K}\left\{\int_0^a \left(\frac{\rho^2 + r^2}{2a^2} - \frac{3}{4}\right) Q(\rho, t)\, \rho\, \mathrm{d}\rho + \int_0^r \ln\left(\frac{a}{r}\right) Q(\rho, t)\, \rho\, \mathrm{d}\rho + \int_r^a \ln\left(\frac{a}{\rho}\right) Q(\rho, t)\, \rho\, \mathrm{d}\rho\right\}. \tag{55.c}$$

If, in addition, $Q(r, t)$ is independent of r, (55.c) gives

$$T_{00}(t) \equiv 0. \tag{55.d}$$

Determination of $T_{01}(r, \varphi, z, t)$

From (19) and (20), with $j = 1$, the differential equation and conditions defining $T_{01}(r, \varphi, z, t)$ are

$$\left.\begin{aligned} &\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right) T_{01}(r, \varphi, z, t) = \frac{1}{\kappa}\frac{\mathrm{d}\Omega_1(t)}{\mathrm{d}t}, \\ &(0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| < b), \end{aligned}\right\} \tag{56}$$

$$-K\frac{\partial T_{01}(r, \varphi, z, t)}{\partial z} = f_1(r, \varphi, t), \quad (0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad z = -b), \tag{57.a}$$

$$K\frac{\partial T_{01}(r, \varphi, z, t)}{\partial z} = 0, \qquad (0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad z = b), \tag{57.b}$$

$$K\frac{\partial T_{01}(r, \varphi, z, t)}{\partial r} = 0, \qquad (r = a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| < b), \tag{57.c}$$

$$\int_0^a \int_0^{2\pi} \int_{-b}^{b} T_{01}(r, \varphi, z, t)\, r\, \mathrm{d}r\, \mathrm{d}\varphi\, \mathrm{d}z = 0 \tag{58}$$

The transformation of (56) and (57) first by (42) and then by (46) leads to the following system:

$$\left.\begin{aligned} &\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \left(\frac{k^2}{r^2} + \frac{n^2\pi^2}{4b^2}\right)\right] \bar{\bar{T}}_{01}(r, k, n, t; \varphi') + \frac{1}{K}\tilde{f}_1(r, k, t; \varphi') = \frac{4\pi b}{\kappa}\, \delta_{k0}\,.\,\delta_{n0}\, \frac{\mathrm{d}\Omega_1(t)}{\mathrm{d}t}, \\ &(0 \leqslant r < a), \end{aligned}\right\} \tag{59}$$

$$K\frac{\partial \bar{\bar{T}}_{01}(r, k, n, t; \varphi')}{\partial r} = 0, \quad (r = a). \tag{60}$$

The solution to the system of (59) and (60) is given by

$$\left.\begin{aligned} &\bar{\bar{T}}_{01}(r, k, n, t; \varphi') = \frac{1}{K}\left[\int_0^r G_{kn}(\rho, r)\, \tilde{f}_1(\rho, k, t; \varphi')\, \rho\, \mathrm{d}\rho + \int_r^a G_{kn}(r, \rho)\, \tilde{f}_1(\rho, k, t; \varphi')\, \rho\, \mathrm{d}\rho\right], \\ &(k = n \neq 0). \end{aligned}\right\} \tag{61.a}$$

For the case of $k = n = 0$, the solution is obtained, after utilizing (58), as

$$\left.\begin{aligned} \bar{\bar{T}}_{01}(r, 0, 0, t; \varphi') = \frac{1}{K}\int_0^{2\pi}\Bigg\{\int_0^a \left(\frac{\rho^2 + r^2}{2a^2} - \frac{3}{4}\right) f_1(\rho, \varphi, t)\, \rho\, \mathrm{d}\rho + \int_0^r \ln\left(\frac{a}{r}\right) f_1(\rho, \varphi, t)\, \rho\, \mathrm{d}\rho \\ + \int_r^a \ln\left(\frac{a}{\rho}\right) f_1(\rho, \varphi, t)\, \rho\, \mathrm{d}\rho\Bigg\}\, \mathrm{d}\varphi. \end{aligned}\right\} \tag{61.b}$$

Introducing (61) into (53) as applied to $T_{01}(r, \varphi, z, t)$, the following expression is obtained:

$$\left.\begin{aligned} T_{01}(r, \varphi, z, t) = {} & \frac{1}{4\pi bK}\int_0^{2\pi}\left\{\int_0^a\left(\frac{\rho^2+r^2}{2a^2}-\frac{3}{4}\right)f_1(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho+\int_0^r \ln\left(\frac{a}{r}\right)f_1(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho\right. \\ & \left.+\int_r^a \ln\left(\frac{a}{\rho}\right)f_1(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho\right\}\mathrm{d}\varphi'+\frac{1}{4\pi bK}\sum_{k=1}^{\infty}\int_0^{2\pi}\left\{\left[\left(\frac{r}{a}\right)^k+\left(\frac{a}{r}\right)^k\right]\int_0^r\left(\frac{\rho}{a}\right)^k f_1(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho\right. \\ & \left.+\left(\frac{r}{a}\right)^k\int_r^a\left[\left(\frac{\rho}{a}\right)^k+\left(\frac{a}{\rho}\right)^k\right]f_1(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho\right\}\frac{\cos k(\varphi-\varphi')}{k}\,\mathrm{d}\varphi' \\ & +\frac{1}{\pi bK}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{\cos(n\pi/2)(1+z/b)}{(1+\delta_{k0})}\int_0^{2\pi}\left[\int_0^r G_{kn}(\rho,r)\,f_1(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho\right. \\ & \left.+\int_r^a G_{kn}(r,\rho)\,f_1(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho\right]\cos k(\varphi-\varphi')\,\mathrm{d}\varphi'. \end{aligned}\right\} \quad (62)$$

For the axisymmetric case (62) reduces to

$$\left.\begin{aligned} T_{01}(r, z, t) = {} & \frac{1}{2bK}\left[\int_0^a\left(\frac{\rho^2+r^2}{2a^2}-\frac{3}{4}\right)f_1(\rho,t)\,\rho\,\mathrm{d}\rho+\int_0^r \ln\left(\frac{a}{r}\right)f_1(\rho,t)\,\rho\,\mathrm{d}\rho\right. \\ & \left.+\int_r^a \ln\left(\frac{a}{\rho}\right)f_1(\rho,t)\,\rho\,\mathrm{d}\rho\right]+\frac{1}{bK}\sum_{n=1}^{\infty}\cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right)\left[\int_0^r G_{0n}(\rho,r)\,f_1(\rho,t)\,\rho\,\mathrm{d}\rho\right. \\ & \left.+\int_r^a G_{0n}(r,\rho)\,f_1(\rho,t)\,\rho\,\mathrm{d}\rho\right]. \end{aligned}\right\} \quad (63.a)$$

If in the above expression $f_1(r, t)$ is independent of r, then in view of

$$\int_0^r G_{0n}(\rho, r)\,\rho\,\mathrm{d}\rho+\int_r^a G_{0n}(r,\rho)\,\rho\,\mathrm{d}\rho=\left(\frac{2b}{\pi n}\right)^2,$$

(63.a) becomes simply

$$T_{01}(z, t)=\left(\frac{2}{\pi}\right)^2\frac{b f_1(t)}{K}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right). \quad (63.b)$$

It is instructive to derive an alternate expression for $T_{01}(r, \varphi, z, t)$ given by (62). For this purpose a finite Hankel transform of order k is defined for $\bar{T}_{01}(r, k, z, t; \varphi')$ as

$$\hat{\bar{T}}_{01}(m, k, z, t; \varphi')=\int_0^a \bar{T}_{01}(r, k, z, t; \varphi')\,J_k(\mu_{km}r)\,r\,\mathrm{d}r. \quad (64)$$

The inversion formula for the above transform is

$$\bar{T}_{01}(r, k, z, t; \varphi')=\sum_{m=0}^{\infty} D_{km}\,\hat{\bar{T}}_{01}(m, k, z, t; \varphi')\,J_k(\mu_{km}r), \quad (65.a)$$

where

$$\frac{1}{D_{km}} = \int_0^a J_k^2(\mu_{km} r)\, r\, \mathrm{d}r = \frac{a^2}{2}\left[1 - \left(\frac{k}{\mu_{km} a}\right)^2\right] J_k^2(\mu_{km} a). \tag{65.b}$$

In (65) it is to be noted that since $\mu_{k0} = 0$ is not an eigenvalue for $k \neq 0$, the summation starts with $m = 1$ for $k \neq 0$. For $k = 0$ the summation starts with $m = 0$, since $\mu_{00} = 0$ is actually an eigenvalue. If now the system of (56) and (57) is transformed by (64), following its transformation by (42), the following system is obtained:

$$\left(\frac{\partial^2}{\partial z^2} - \mu_{km}^2\right)\hat{\hat{T}}_{01}(m, k, z, t; \varphi') = \frac{\pi a^2}{\kappa}\, \delta_{k0} \,.\, \delta_{m0} \frac{\mathrm{d}\Omega_1(t)}{\mathrm{d}t}, \qquad (|z| < b), \tag{66}$$

$$-K\frac{\partial \hat{\hat{T}}_{01}(m, k, z, t; \varphi')}{\partial z} = \hat{\hat{f}}_1(m, k, t; \varphi'), \qquad (z = -b), \tag{67.a}$$

$$K\frac{\partial \hat{\hat{T}}_{01}(m, k, z, t; \varphi')}{\partial z} = 0, \qquad (z = b). \tag{67.b}$$

In terms of transform notation (58) can be expressed as

$$\int_{-b}^{b} \hat{\hat{T}}_{01}(0, 0, z, t; \varphi')\, \mathrm{d}z = 0. \tag{68}$$

The solution to the system of (66) and (67) is given by

$$\hat{\hat{T}}_{01}(m, k, z, t; \varphi') = \frac{\cosh \mu_{km}(b - z)}{K \mu_{km} \sinh(2\mu_{km} b)}\, \hat{\hat{f}}_1(m, k, t; \varphi'), \qquad (k = m \neq 0). \tag{69.a}$$

For the case of $\mu_{km} = \mu_{00} = 0$, the solution is obtained, after utilizing (68), as

$$\hat{\hat{T}}_{01}(0, 0, z, t; \varphi') = \frac{1}{Kb}\left[\frac{(z - b)^2}{4} - \frac{b^2}{3}\right] \hat{\hat{f}}_1(0, 0, t; \varphi'). \tag{69.b}$$

The result of introducing (69) into the combination of (65) and (42) as applied to $T_{01}(r, \varphi, z, t)$, is

$$T_{01}(r, \varphi, z, t) = \frac{1}{\pi a^2 b K}\left[\frac{(b - z)^2}{4} - \frac{b^2}{3}\right] \int_0^a \int_0^{2\pi} f_1(r, \varphi, t)\, r\, \mathrm{d}r\, \mathrm{d}\varphi$$

$$+ \frac{2}{\pi a K} \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \frac{J_k(\mu_{km} r) \cosh[\mu_{km}(b - z)] \int_0^a \int_0^{2\pi} f_1(r, \varphi', t)\, J_k(\mu_{km} r) \cos k(\varphi - \varphi')\, r\, \mathrm{d}r\, \mathrm{d}\varphi'}{(1 + \delta_{k0})\,[1 - (k/\mu_{km} a)^2]\,(\mu_{km} a)\, J_k^2(\mu_{km} a) \sinh(2\mu_{km} b)}. \tag{70}$$

Thus, (70) and (62) comprise two solution forms for $T_{01}(r, \varphi, z, t)$. They are both in the form of double infinite series. For purposes of numerical computation, (62) is preferred to (70) since the latter requires the solution of (35) for the eigenvalues μ_{km}, whereas (62) employs summations over positive integer values of k and n. A similar alternate expression can be obtained for $T_{00}(r, \varphi, z, t)$ in the same way. If $f_1(r, \varphi, t)$ is independent of φ, (70) yields the axisymmetric case

$$T_{01}(r, z, t) = \frac{2}{a^2bK}\left[\frac{(b-z)^2}{4} - \frac{b^2}{3}\right]\int_0^a f_1(r, t)\, r\, \mathrm{d}r$$

$$+ \frac{2}{aK}\sum_{m=1}^{\infty}\frac{J_0(\mu_m r)\cosh[\mu_m(b-z)]\int_0^a f_1(r, t)\, J_0(\mu_m r)\, r\, \mathrm{d}r}{(\mu_m a)\, J_0^2(\mu_m a)\sinh(2\mu_m b)}, \tag{71.a}$$

where the subscript $k = 0$ in μ_{km} has been dropped. This expression should be compared with (63.a). If $f_1(r, t)$ is independent of r, then since

$$\int_0^a J_0(\mu_m r)\, r\, \mathrm{d}r = 0, \qquad (m \neq 0),$$

(71.a) reduces to

$$T_{01}(z, t) = \frac{f_1(t)}{bK}\left[\frac{(b-z)^2}{4} - \frac{b^2}{3}\right]. \tag{71.b}$$

From (71.b) and (63.b) it follows that

$$\frac{(b-z)^2}{4} - \frac{b^2}{3} = \left(\frac{2b}{\pi}\right)^2 \sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{2}\left(1 + \frac{z}{b}\right), \tag{72}$$

which is the well-known Fourier expansion.

Determination of $T_{02}(r, \varphi, z, t)$

From (19) and (20), with $j = 2$, the differential equation and conditions defining $T_{02}(r, \varphi, z, t)$ are

$$\left.\begin{aligned}&\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right) T_{02}(r, \varphi, z, t) = \frac{1}{\kappa}\frac{\mathrm{d}\Omega_2(t)}{\mathrm{d}t},\\ &(0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| < b),\end{aligned}\right\} \tag{73}$$

$$-K\frac{\partial T_{02}(r, \varphi, z, t)}{\partial z} = 0, \qquad (0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad z = -b), \tag{74.a}$$

$$K\frac{\partial T_{02}(r, \varphi, z, t)}{\partial z} = f_2(r, \varphi, t), \quad (0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad z = b), \tag{74.b}$$

$$K\frac{\partial T_{02}(r, \varphi, z, t)}{\partial r} = 0, \qquad (r = a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| < b), \tag{74.c}$$

$$\int_0^a \int_0^{2\pi} \int_{-b}^{b} T_{02}(r, \varphi, z, t)\, r\, \mathrm{d}r\, \mathrm{d}\varphi\, \mathrm{d}z = 0. \tag{75}$$

The solution to the system of (73), (74) and (75) is obtained in exactly the same fashion as that used for $T_{01}(r, \varphi, z, t)$. The result is given by

$$\left.\begin{aligned}
&T_{02}(r,\varphi,z,t)=\frac{1}{4\pi bK}\int_0^{2\pi}\left[\int_0^a\left(\frac{\rho^2-r^2}{2a^2}-\frac{3}{4}\right)f_2(\rho,\varphi,t)\rho\,\mathrm{d}\rho+\int_0^r\ln\left(\frac{a}{r}\right)f_2(\rho,\varphi,t)\,\rho\,\mathrm{d}\rho+\right.\\
&\left.\int_r^a\ln\left(\frac{a}{\rho}\right)f_2(\rho,\varphi,t)\,\rho\,\mathrm{d}\rho\right]\mathrm{d}\varphi+\frac{1}{4\pi bK}\sum_{k=1}^{\infty}\int_0^{2\pi}\left\{\left[\left(\frac{r}{a}\right)^k+\left(\frac{a}{r}\right)^k\right]\int_0^r\left(\frac{\rho}{a}\right)^k f_2(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho+\right.\\
&\left.\left(\frac{r}{a}\right)^k\int_r^a\left[\left(\frac{\rho}{a}\right)^k+\left(\frac{a}{\rho}\right)^k\right]f_2(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho\right\}\frac{\cos k(\varphi-\varphi')}{k}\,\mathrm{d}\varphi'+\\
&\frac{1}{\pi bK}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi/2)(1+z/b)}{(1+\delta_{k0})}\int_0^{2\pi}\left[\int_0^r G_{kn}(\rho,r)\,f_2(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho+\right.\\
&\left.\int_r^a G_{kn}(r,\rho)\,f_2(\rho,\varphi',t)\,\rho\,\mathrm{d}\rho\right]\cos k(\varphi-\varphi')\,\mathrm{d}\varphi'.
\end{aligned}\right\}\quad(76)$$

For the axisymmetric case (76) gives

$$T_{02}(r,z,t)=\frac{1}{2bK}\left[\int_0^a\left(\frac{\rho^2+r^2}{2a^2}-\frac{3}{4}\right)f_2(\rho,t)\,\rho\,\mathrm{d}\rho+\int_0^r\ln\left(\frac{a}{r}\right)f_2(\rho,t)\,\mathrm{d}\rho+\int_r^a\ln\left(\frac{a}{\rho}\right)f_2(\rho,t)\,\rho\,\mathrm{d}\rho\right]$$
$$+\frac{1}{bK}\sum_{n=1}^{\infty}(-1)^n\cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right)\left[\int_0^r G_{0n}(\rho,r)\,f_2(\rho,t)\,\rho\,\mathrm{d}\rho+\int_r^a G_{0n}(r,\rho)\,f_2(\rho,t)\,\rho\,\mathrm{d}\rho\right].\quad(77.\mathrm{a})$$

In the one-dimensional case in the z-co-ordinate, (77.a) reduces to

$$T_{02}(z,t)=\left(\frac{2}{\pi}\right)^2\frac{bf_2(t)}{K}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right).\quad(77.\mathrm{b})$$

Similarly, through the application of the finite Hankel transform of (64), the following alternate expression is obtained for $T_{02}(r,\varphi,z,t)$:

$$T_{02}(r,\varphi,z,t)=\frac{1}{\pi a^2bK}\left[\frac{(b+z)^2}{4}-\frac{b^2}{3}\right]\int_0^a\int_0^{2\pi}f_2(r,\varphi,t)\,r\,\mathrm{d}r\,\mathrm{d}\varphi+\frac{2}{\pi aK}\sum_{k=0}^{\infty}\sum_{m=1}^{\infty}$$
$$\frac{J_k(\mu_{km}r)\cosh[\mu_{km}(b+z)]\int_0^a\int_0^{2\pi}f_2(r,\varphi',t)\,J_k(\mu_{km}r)\cos k(\varphi-\varphi')\,r\,\mathrm{d}r\,\mathrm{d}\varphi'}{(1+\delta_{k0})[1-(k/\mu_{km}a)^2](\mu_{km}a)\,J_k^2(\mu_{km}a)\sinh(2\mu_{km}b)}.\quad(78)$$

The axisymmetric case is readily obtained from (78) as

$$T_{02}(r,z,t)=\frac{2}{a^2bK}\left[\frac{(b+z)^2}{4}-\frac{b^2}{3}\right]\int_0^a f_2(r,t)\,r\,\mathrm{d}r$$
$$+\frac{2}{aK}\sum_{m=1}^{\infty}\frac{J_0(\mu_m r)\cosh[\mu_m(b+z)]\int_0^a f_2(r,t)\,J_0(\mu_m r)\,r\,\mathrm{d}r}{(\mu_m a)\,J_0^2(\mu_m a)\sinh(2\mu_m b)},\quad(79.\mathrm{a})$$

and the one-dimensional case follows from (79.a) as

$$T_{02}(z, t) = \frac{f_2(t)}{bK}\left[\frac{(b+z)^2}{4} - \frac{b^2}{3}\right]. \tag{79.b}$$

The comparison of (77.b) and (79.b) gives the Fourier expansion

$$\frac{(b+z)^2}{4} - \frac{b^2}{3} = \left(\frac{2b}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right), \tag{80}$$

which should be compared with (72).

Determination of $T_{03}(r, \varphi, z, t)$

From (19) and (20), with $j = 3$, the differential equation and conditions defining $T_{03}(r, \varphi, z, t)$ are

$$\left.\begin{aligned}&\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}\right) T_{03}(r, \varphi, z, t) = \frac{1}{\kappa}\frac{\mathrm{d}\Omega_3(t)}{\mathrm{d}t},\\ &(0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| < b),\end{aligned}\right\} \tag{81}$$

$$-K\frac{\partial T_{03}(r, \varphi, z, t)}{\partial z} = 0, \qquad (0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad z = -b), \tag{82.a}$$

$$K\frac{\partial T_{03}(r, \varphi, z, t)}{\partial z} = 0, \qquad (0 \leqslant r < a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad z = b), \tag{82.b}$$

$$K\frac{\partial T_{03}(r, \varphi, z, t)}{\partial r} = f_3(\varphi, z, t), \quad (r = a, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad |z| < b), \tag{82.c}$$

$$\int_0^a \int_0^{2\pi} \int_{-b}^{b} T_{03}(r, \varphi, z, t)\, r\, \mathrm{d}r\, \mathrm{d}\varphi\, \mathrm{d}z = 0. \tag{83}$$

As in the case of $T_{00}(r, \varphi, z, t)$ and $T_{01}(r, \varphi, z, t)$, the system of (81) and (82) is transformed successively by (42) and (46). The solution of the resulting system when introduced into (53) as applied to $T_{03}(r, \varphi, z, t)$ yields the expression

$$\left.\begin{aligned}T_{03}(r, \varphi, z, t) = {}& \frac{1}{8\pi K}\left(\frac{a}{b}\right)\left(\frac{r^2}{a^2} - \frac{1}{2}\right)\int_0^{2\pi}\int_{-b}^{b} f_3(\varphi, z, t)\,\mathrm{d}\varphi\,\mathrm{d}z + \frac{1}{2\pi K}\left(\frac{a}{b}\right)\sum_{k=1}^{\infty}\left(\frac{r}{a}\right)^k\\ &\int_0^{2\pi}\int_{-b}^{b} f_3(\varphi', z, t)\frac{\cos k(\varphi - \varphi')}{k}\,\mathrm{d}\varphi'\,\mathrm{d}z + \frac{2}{\pi^2 K}\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\frac{I_k(n\pi r/2b)}{I'_k(n\pi a/2b)}\cdot\frac{\cos(n\pi/2)(1 + z/b)}{n(1 + \delta_{k0})}\\ &\int_0^{2\pi}\int_{-b}^{b} f_3(\varphi', z, t)\cos k(\varphi - \varphi')\cos\frac{n\pi}{2}\left(1 + \frac{z}{b}\right)\mathrm{d}\varphi'\,\mathrm{d}z.\end{aligned}\right\} \tag{84}$$

In the case of axial symmetry, (84) reduces to

$$\left.\begin{aligned}T_{03}(r, z, t) = {}& \frac{1}{4K}\left(\frac{a}{b}\right)\left(\frac{r^2}{a^2} - \frac{1}{2}\right)\int_{-b}^{b} f_3(z, t)\,\mathrm{d}z + \frac{2}{\pi K}\sum_{n=1}^{\infty}\frac{I_0(n\pi r/2b)}{I'_0(n\pi a/2b)}\cdot\frac{\cos(n\pi/2)(1 + z/b)}{n}\\ &\int_{-b}^{b} f_3(z, t)\cos\frac{n\pi}{2}\left(1 + \frac{z}{b}\right)\mathrm{d}z.\end{aligned}\right\} \tag{85.a}$$

If $f_3(\varphi, z, t)$ is independent of z, there is no axial conduction and (84) gives

$$T_{03}(r, \varphi, t) = \frac{a}{4\pi K}\left(\frac{r^2}{a^2} - \frac{1}{2}\right) \int_0^{2\pi} f_3\,(\varphi t)\,\mathrm{d}\varphi + \frac{a}{\pi K} \sum_{k=1}^{\infty} \left(\frac{r}{a}\right)^k \int_0^{2\pi} f_3(\varphi', t)\,\frac{\cos k(\varphi - \varphi')}{k}\,\mathrm{d}\varphi'. \tag{85.b}$$

If $f_3(\varphi, z, t)$ is independent of both φ and z, the problem becomes one-dimensional in the radial direction, and (85.a) and (85.b) both reduce to

$$T_{03}\,(r, t) = \frac{a}{2K}\left(\frac{r^2}{a^2} - \frac{1}{2}\right) f_3(t). \tag{85.c}$$

An expression alternate to (84) can be obtained for $T_{03}(r, \varphi, z, t)$ as in the case of $T_{01}(r, \varphi, z, t)$ and $T_{02}(r, \varphi, z, t)$.

This concludes the determination of the $T_{0j}(r, \varphi, z, t)$ functions appearing in (38). In summary, $T_{00}(r, \varphi, z, t)$ is given by (54); $T_{01}(r, \varphi, z, t)$ by either (62) or (70); $T_{02}(r, \varphi, z, t)$ by either (76) or (78); and $T_{03}(r, \varphi, z, t)$ by (84).

Numerous special cases of heat-conduction problems, with boundary conditions of the second kind, follow from the general solutions (38). As an example, suppose that there is constant flux at $r = a$, the faces $z = \pm b$ are insulated, there is no heat generation throughout the volume, and the initial temperature is zero. Then, from (37.d),

$$\Omega_3(t) = \frac{2f_3\kappa t}{Ka},$$

and from (85.c),

$$T_{03}(r) = \frac{af_3}{2K}\left(\frac{r^2}{a^2} - \frac{1}{2}\right).$$

In (38.b) the only non-vanishing terms are those corresponding to $k = n = 0$, and (38.b) reduces to

$$T(r, t) = \Omega_3(t) + T_{03}(r) - \frac{2f_3}{Ka} \sum_{m=1}^{\infty} \frac{\exp\,(-\kappa\mu_m^2 t)}{\mu_m^2} \cdot \frac{J_0(\mu_m r)}{J_0(\mu_m a)},$$

or

$$T(r, t) = \frac{2af_3}{K}\left[\frac{\kappa t}{a^2} + \frac{1}{4}\left(\frac{r^2}{a^2} - \frac{1}{2}\right) - \sum_{m=1}^{\infty} \frac{\exp\,(-\kappa\mu_m^2 t)}{(\mu_m a)^2} \cdot \frac{J_0(\mu_m r)}{J_0(\mu_m a)}\right], \tag{86}$$

where, in view of (35.a), the eigenvalues μ_m are determined from the positive roots of

$$-J_0'(\mu_m a) = J_1(\mu_m a) = 0.$$

Expression (86) is given as equation (1) on p. 203 in [4].

Another example that has been studied by several authors [5], [6], [7] is the one-dimensional flow of heat in a slab one face of which is subjected to a heat flux given as an arbitrary function of time, when the other face is insulated. To obtain this case from the general expression (38.b), it is sufficient to let $F(r, \varphi, z) = F = \text{const.}$, $Q(r, \varphi, z, t) = 0$, $f_1(r, \varphi, t) = 0$, $f_2(r, \varphi, t) = f_2(t)$, $f_3(\varphi, z, t) = 0$ in

(38.b), whereby the only non-vanishing terms are those corresponding to $k = m = 0$, that is $\mu_{km} = \mu_{00} = 0$. Thus, (38.b) yields

$$T(z,t) = F + \Omega_2(t) + T_{02}(z,t) - \frac{4b}{\pi^2 K}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right)\exp\left(-\frac{n^2\pi^2}{4b^2}\kappa t\right)\left[f_2(0) + \int_0^t \exp\left(\frac{n^2\pi^2}{4b^2}\kappa\tau\right) f_2'(\tau)\,\mathrm{d}\tau\right].$$

From (37.c),

$$\Omega_2(t) = \frac{\kappa}{2bK}\int_0^t f_2(\tau)\,\mathrm{d}\tau,$$

and $T_{02}(z,t)$ is given by (79.b). Thus,

$$\left.\begin{aligned} T(z,t) = F &+ \frac{\kappa}{2bK}\int_0^t f_2(\tau)\,\mathrm{d}\tau + \left[\frac{(b+z)^2}{4} - \frac{b^2}{3}\right]\frac{f_2(t)}{bK} \\ &- \frac{4b}{\pi^2 K}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right)\exp\left(-\frac{n^2\pi^2}{4b^2}\kappa t\right)\left[f_2(0) + \int_0^t \exp\left(\frac{n^2\pi^2}{4b^2}\kappa\tau\right) f_2'(\tau)\,\mathrm{d}\tau\right]. \end{aligned}\right\} \quad (87)$$

The equivalence of this expression and equation (20) or (21) of [5] follows from the substitution $x = b + z = (a/2) + z$. If the heating rate is independent of time, and $F = 0$, (87) becomes

$$T(z,t) = \frac{bf_2}{K}\left\{\frac{\kappa t}{2b^2} + \left[\frac{1}{4}\left(1+\frac{z}{b}\right)^2 - \frac{1}{3}\right] - \left(\frac{2}{\pi}\right)^2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right)\exp\left(-\frac{n^2\pi^2}{4b^2}\kappa t\right)\right\}, \quad (88)$$

which corresponds to equation (3) on p. 112 in [4]. These one-dimensional solutions, (86), (87) and (88), could have been obtained directly from the general solution (22.b) along with the use of (10), (16), (19) and (20).

A further example which has been treated in [8] in connection with the heating of semiconductor devices is the one-dimensional unsteady temperature distribution in a thin, thermally insulated disk initially at zero temperature and suddenly heated by a heat source of uniform volume density in a circular area at the center. The solution given as the combination of equations (5) and (6) of [8] is not very suitable for purposes of numerical evaluation. This is due to the presence of the factor, $[1 - \exp(-\tau\lambda^2)]$, in the series summation, which introduces the poor convergence. The solution to this problem expressed in a more suitable form can be readily obtained from the general expression (38.b). Indeed, letting $F = f_1 = f_2 = f_3 = 0$, and $Q = Q_0[H(r) - H(r - \eta a)]$ in (38.b), where $H(x)$ is the Heaviside unit step function and ηa $(0 \leqslant \eta \leqslant 1)$ is the radius of the circle in which the constant heat source Q_0 is acting, yields the expression

$$T(r,t) = \Omega_0(t) + T_{00}(r) - 2\frac{Q_0}{K}a^2\eta\sum_{m=1}^{\infty}\frac{J_1(\mu_m a\eta)J_0(\mu_m r)}{J_0^2(\mu_m a)}\cdot\frac{\exp(-\kappa\mu_m^2 t)}{(\mu_m a)^3}, \quad (89)$$

after evaluating the integral $\int_0^{\eta a} J_0(\mu_m r)\, r\,\mathrm{d}r$. From (37.a) and the above-mentioned expression for Q, it follows that

$$\Omega_0(t) = \frac{Q_0}{K}\eta^2\kappa t. \quad (90)$$

Using this value of $\Omega_0(t)$, $T_{00}(r)$ is readily determined from the one-dimensionalized version of (39), (40) and (41) as

$$T_{00}(r) = \frac{Q_0\, a^2}{4K} \cdot \begin{Bmatrix} \left[2\eta^2 \ln\left(\frac{1}{\eta}\right) - (1-\eta^2)\left(\frac{r^2}{a^2}+\frac{1}{2}\eta^2\right)\right], & 0 \leqslant \frac{r}{a} \leqslant \eta, \\ \left[2\eta^2 \ln\left(\frac{a}{r}\right) + \eta^2\left(\frac{r^2}{a^2}+\frac{3}{2}\eta^2-\frac{1}{2}\right)\right], & \eta \leqslant \frac{r}{a} \leqslant 1. \end{Bmatrix} \tag{91}$$

The expression (89) used in combination with (90) and (91) constitutes the solution, and $\mu_m a$ is the m^{th} positive root of $J_1(\mu_m a) = 0$. With $j = 0$, it follows from (21) specialized for the one-dimensional case in r, that

$$\frac{1}{8\eta} \begin{Bmatrix} \left[2\eta^2 \ln\left(\frac{1}{\eta}\right) - (1-\eta^2)\left(\frac{r^2}{a^2}+\frac{1}{2}\eta^2\right)\right], & 0 \leqslant \frac{r}{a} \leqslant \eta, \\ \left[2\eta^2 \ln\left(\frac{a}{r}\right) + \eta^2\left(\frac{r^2}{a^2}+\frac{1}{2}\eta^2-\frac{3}{2}\right)\right], & \eta \leqslant \frac{r}{a} \leqslant 1 \end{Bmatrix} = \sum_{m=1}^{\infty} \frac{J_1(\mu_m a\eta)\, J_0(\mu_m r)}{(\mu_m a)^3 J_0^2(\mu_m a)}, \quad 0 \leqslant r \leqslant a. \tag{92}$$

The equivalence of the present solution given by (89) and that given by the combination of (5) and (6) of [8] follows from the summation formula (92). The difference between the two solutions lies in the fact that, (89) utilizes the left-hand side of (92), an expression in closed-form, whereas the combination of (5) and (6) of [8] utilizes the right-hand side of (92) which is a slowly converging series expression.

Lastly, a set of two one-dimensional problems, one for the cylinder and the other for the slab is considered and numerical results are presented in the form of charts. Consider a cylinder insulated at the faces $z = \pm b$ and subjected at the surface $r = a$ to a heat flux proportional to time, the initial temperature and internal heat source being zero. In this case $f_3(\varphi, z, t) = f_3 t$ in (37) and (84) which yield, respectively,

$$\Omega_3(t) = \frac{f_3\, a^3}{K\kappa}\left(\frac{\kappa t}{a^2}\right)^2$$

$$T_{03}(r, t) = \frac{f_3\, a^3}{2K\kappa}\left(\frac{r^2}{a^2}-\frac{1}{2}\right)\cdot\frac{\kappa t}{a^2}.$$

Furthermore, $F = f_1 = f_2 = Q = 0$ and $f_3(\varphi, z, t) = f_3 t$ in (38.b) which becomes

$$T(r, t) = \Omega_3(t) + T_{03}(r, t) - 2\frac{f_3 a}{K}\sum_{m=1}^{\infty}\frac{J_0(\mu_m r)}{J_0(\mu_m a)}\cdot\frac{\exp(-\kappa\mu_m^2 t)}{(\mu_m a)^2}\int_0^t \exp(\kappa\mu_m^2\tau)\,d\tau.$$

Performing the indicated integration in the above expression and making use of the summation formula

$$\frac{1}{32}\left(\frac{r^2}{a^2}-\frac{1}{2}\frac{r^4}{a^4}-\frac{1}{3}\right) = \sum_{m=1}^{\infty}\frac{J_0(\mu_m r)}{J_0(\mu_m a)}\cdot\frac{1}{(\mu_m a)^4}, \quad 0 \leqslant r \leqslant a, \tag{93}$$

one obtains

$$\left(\frac{K\kappa}{f_3 a^3}\right) T(r, t) = \left(\frac{\kappa t}{a^2}\right)^2 + \frac{1}{2}\left(\frac{r^2}{a^2}-\frac{1}{2}\right)\cdot\frac{\kappa t}{a^2} - \frac{1}{16}\left(\frac{r^2}{a^2}-\frac{1}{2}\frac{r^4}{a^4}-\frac{1}{3}\right) + 2\sum_{m=1}^{\infty}\frac{J_0(\mu_m r)}{J_0(\mu_m a)}\cdot\frac{\exp(-\mu_m^2\kappa t)}{(\mu_m a)^4}, \tag{94}$$

where $\mu_m a$ is the m^{th} positive root of $J_1(\mu_m a) = 0$. For large values of t the infinite series part of (94) becomes vanishingly small and the first three groups of terms of (94) correspond to the quasi-steady value of $T(r, t)$. Using the tables of Bessel functions given in [9] and the roots of $J_1(\mu_m a) = 0$ tabulated in [2], the sum of the third group of terms and the infinite series appearing on the right-hand side of (94) has been numerically evaluated. These results are presented graphically in Fig. 1 for various values of the Fourier number, $(\kappa t/a^2)$, in which the ordinates give the values of

$$\left(\frac{K\kappa}{f_3\, a^3}\right) T(r, t) - \left(\frac{\kappa t}{a^2}\right)^2 - \frac{1}{2}\left(\frac{r^2}{a^2} - \frac{1}{2}\right)\frac{\kappa t}{a^2}.$$

Similarly, the corresponding results are readily obtained for the infinite slab $|z| \leqslant b$ with the face $z = b$ subjected to a heat flux varying linearly with time while the face $z = -b$ is insulated. Letting $F = 0$ and $f_2(t) = f_2 t$ in (87), carrying out the integration and utilizing the summation formula

$$\frac{1}{96}\left[\left(1 + \frac{z}{b}\right)^2 - \frac{1}{8}\left(1 + \frac{z}{b}\right)^4 - \frac{14}{15}\right] = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n\pi)^4} \cos\frac{n\pi}{2}\left(1 + \frac{z}{b}\right), \quad |z| \leqslant b, \tag{95}$$

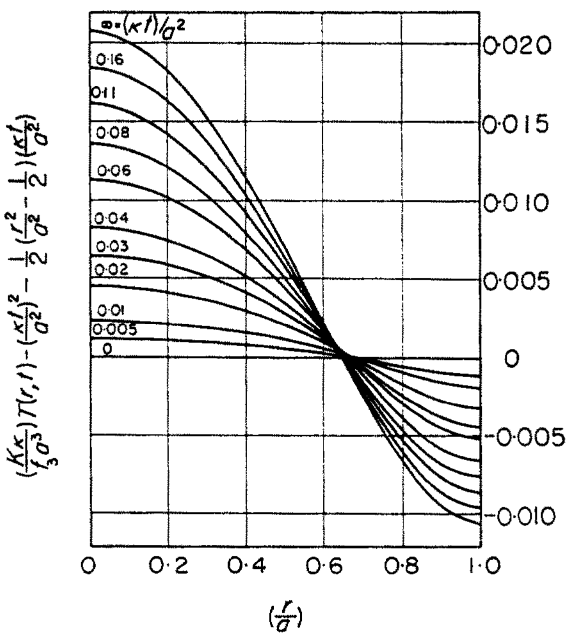

FIG. 1. Temperature response charts for the cylinder, $0 < r < a$, based on equation (94).

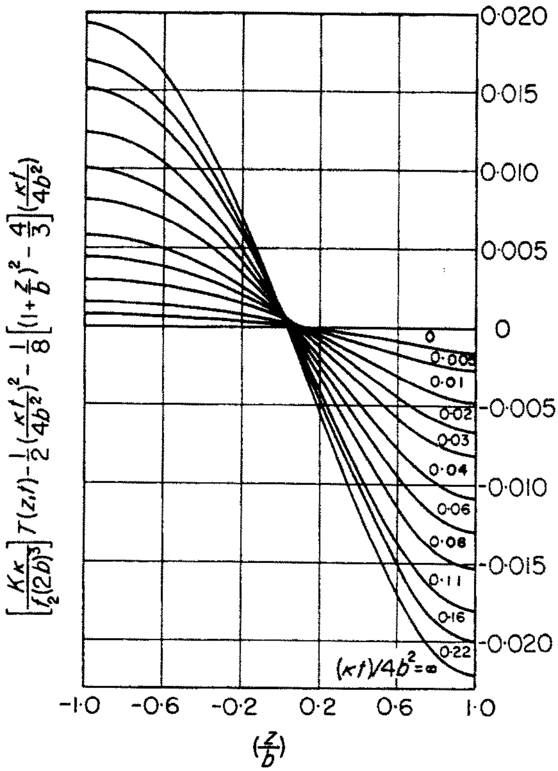

FIG. 2. Temperature response charts for the slab, $|z| < b$ based on equation (96).

the following expression is obtained:

$$\left[\frac{K\kappa}{f_2(2b)^3}\right] T(z,t) = \frac{1}{2}\left(\frac{\kappa t}{4b^2}\right)^2 + \frac{1}{8}\left[\left(1+\frac{z}{b}\right)^2 - \frac{4}{3}\right]\cdot\frac{\kappa t}{4b^2} - \frac{1}{48}\left[\left(1+\frac{z}{b}\right)^2 - \frac{1}{8}\left(1+\frac{z}{b}\right)^4 - \frac{14}{15}\right]$$
$$+ 2\sum_{n=1}^{\infty}(-1)^n \cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right)\cdot\frac{\exp\left(-n^2\pi^2\kappa t/4b^2\right)}{(n\pi)^4}, \tag{96}$$

which should be compared with (94). The quasi-steady response is described by the first three groups of terms appearing on the right-hand side of (96). Graphs of

$$\left[\frac{K\kappa}{f_2(2b)^3}\right] T(z,t) - \frac{1}{2}\left(\frac{\kappa t}{4b^2}\right)^2 - \frac{1}{8}\left[\left(1+\frac{z}{b}\right)^2 - \frac{4}{3}\right]\cdot\frac{\kappa t}{4b^2}$$

versus (z/b) for various values of the Fourier number, $(\kappa t/4b^2)$, are presented in Fig. 2.

CONCLUDING REMARKS

The example of the cylinder problem treated here as an application of the general method shows that heat-conduction problems with boundary conditions of the second kind, no matter how complicated they may be, can be solved directly by use of the expressions (22), provided that the eigenvalue problem defined by (10) is solvable. The method does not require the use of Duhamel's Superposition Theorem and is not restricted to any particular form of the geometry of region. The usual method of treating conductive heat-transfer problems with boundary conditions of the second kind is the Laplace transform technique. Especially in the case of complicated problems, the difficulties inherent in and the excessive amount of labor required for the Laplace inversion procedure are well-known. This is one feature of the Laplace transform technique that prohibits its practical application to general problems. Another limitation of this technique is the fact that it necessitates that the geometry of the region be given in advance, thus making it impossible to obtain a generalized and unified treatment with respect to geometry. The present method eliminates these difficulties and supplies the solution directly.

Finally, it is worthwhile to note that (22.b) could have been expressed in the form of

$$T(P,t) = \frac{1}{V}\int_R F(P)\,\mathrm{d}V + \frac{\kappa}{KV}\int_0^t \left[\int_R Q(P,\tau)\,\mathrm{d}V + \sum_{i=1}^{q}\int_{S_i} f_i(s_i,\tau)\,\mathrm{d}S_i\right]\mathrm{d}\tau$$
$$+ \sum_{m=1}^{\infty} C_m\,\phi_m(P)\exp\left(-\kappa\lambda_m^2 t\right)\left\{\int_R \phi_m(P)\,F(P)\,\mathrm{d}V + \right.$$
$$\left. + \frac{\kappa}{K}\int_0^t \exp\left(\kappa\lambda_m^2\tau\right)\left[\int_R \phi_m(P)\,Q(P,\tau)\,\mathrm{d}V + \sum_{i=1}^{q}\int_{S_i}\phi_m(s_i)\,f_i(s_i,\tau)\,\mathrm{d}S_i\right]\mathrm{d}\tau\right\}, \tag{97}$$

in which the pseudo-steady functions $T_{0j}(P,t)$ do not appear. Although (97) is simpler than (22), the infinite series part of it does not converge uniformly but only in the Fourier sense. To illustrate this point further, consider the simple problem, the solution to which is expressed by (87). Direct application of (97) to this problem gives the result

$$T(z,t) = F + \frac{\kappa}{2bK}\int_0^t f_2(\tau)\,\mathrm{d}\tau + \frac{\kappa}{Kb}\sum_{n=1}^{\infty}(-1)^n\cos\frac{n\pi}{2}\left(1+\frac{z}{b}\right)\exp\left(-n^2\pi^2\frac{\kappa t}{4b^2}\right)\int_0^t \exp\left(n^2\pi^2\frac{\kappa\tau}{4b^2}\right)f_2(\tau)\,\mathrm{d}\tau. \tag{98}$$

This should be compared with (87). The expression (98) corresponds to equation (8) of [6], except for the sign of the first term which has been corrected and expressed by equation (16) of [7].

It is interesting to note that the summations $\sum_{k=1}^{\infty}$ appearing in (54), (55.b), (62), (76), (84) and (85.b) can be expressed in closed form by means of the following summation formula:

$$\sum_{k=1}^{\infty}\left(\frac{r}{a}\right)^k\frac{\cos k(\varphi-\varphi')}{k} = -\frac{1}{2}\ln\left[1-2\left(\frac{r}{a}\right)\cos(\varphi-\varphi')+\left(\frac{r}{a}\right)^2\right].$$

Thus, for example, (85.b) becomes

$$T_{03}(r,\varphi,t) = \frac{a}{4\pi K}\left(\frac{r^2}{a^2}-\frac{1}{2}\right)\int_0^{2\pi} f_3(\varphi,t)\,\mathrm{d}\varphi - \frac{a}{2\pi K}\int_0^{2\pi} f_3(\varphi',t)\ln\left[1-2\left(\frac{r}{a}\right)\cos(\varphi-\varphi')+\left(\frac{r}{a}\right)^2\right]\mathrm{d}\varphi'. \tag{99}$$

However, the resulting integrals are, in general, more complicated to evaluate.

REFERENCES

1. N. Y. Ölçer, On the theory of conductive heat transfer in finite regions, *Int. J. Heat Mass Transfer* **7**, 307–314 (1964).
2. F. W. J. Olver, ed., Bessel Functions, Part III, zeros and associated values, *Royal Society Mathematical Tables*, Vol. 7. Cambridge University Press, Cambridge (1960).
3. G. W. Morgenthaler and H. Reismann, Zeros of first derivatives of Bessel Functions of the first kind, $J_n'(x)$, $21 < n < 51$, $0 < x < 100$, *J. Res. Nat. Bur. Stand., Wash.* **67B**, 181–183 (1963).
4. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, 2nd Ed. Clarendon Press, Oxford (1959).
5. S.-Y. Chen, One-dimensional heat conduction with arbitrary heating rate, *J. Aerospace Sci.* (Readers' Forum) **28**, 336–337 (1961).
6. P. D. Thomas, Comment on one-dimensional heat conduction with arbitrary heating rate, *J. Aerospace Sci.* (Readers' Forum) **29**, 616–617 (1962).
7. I. U. Ojalvo, M. Newman and M. Forray, Another comment on one-dimensional heat conduction with arbitrary heating rate, *J. Aerospace Sci.* (Readers' Forum) **29**, 1126–1127 (1962).
8. T. D. Riney, Disk heated by internal source, *Trans. Amer. Soc. Mech. Engrs, J. Appl. Mech.* **83**, Ser. E, 631–632 (1961).
9. Harvard Computation Laboratory, *Tables of the Bessel Functions of the first kind of orders* 0 *through* 135, Vol. 1. Harvard University Press, Cambridge (1947).

Résumé—Des expressions générales sont obtenues pour des distributions de températures transitoires dans des régions finies de géométrie arbitraire, sous des conditions de flux de chaleur imposé sur toutes les frontières et avec des sources de chaleur dépendant du temps et des conditions initiales arbitraires. Les sources (ou les puits) de chaleur sont distribuées dans l'espace et peuvent comme cas particuliers, être des sources surfaciques, linéiques ou ponctuelles. En introduisant en plus certaines fonctions de source de chaleur fictives, des solutions pseudo-permanentes correspondantes sont définies, au moyen desquelles les champs de température sont exprimés sous forme de solutions en série uniformément convergentes. La méthode générale de solution est appliquée à une étude détaillée d'un problème de cylindre fini de nautre très générale, qui n'a pas été traité auparavant.

L'étude actuelle constitue un complément d'un article antérieur dans lequel on fait l'hypothèse de l'existence de solutions permanentes lorsque les fonctions de source volumique et surfacique imposées sont indépendantes du temps.

Zusammenfassung—Für instationäre Temperaturverteilungen in endlichen Bereichen beliebiger Geometrie werden allgemeine Ausdrücke abgeleitet unter der Bedingung einer vorgeschriebenen Wärmestromdichte an allen Abgrenzungen und mit zeitabhängigen Wärmequellen und beliebigen Anfangszuständen. Die Wärmequellen (oder Senken) sind über das ganze Volumen verteilt und können, wie in speziellen Fällen, Flächen-, Linien- oder Punktquellen sein.

Durch Einführen bestimmter künstlicher zusätzlicher Funktionen für die Wärmequellen werden entsprechende pseudo-stationäre Lösungen definiert, mit deren Hilfe die Temperaturfelder in Form von gleichförmig konvergenten Reihenlösungen ausgedrückt werden. Die allgemeine Lösungsmethode wird auf eine eingehende Studie eines endlichen Zylinderproblems ziemlich allgemeiner Natur, welches bisher nicht behandelt worden ist, angewandt. Die vorliegende Arbeit ergänzt nachträglich einen früheren Aufsatz, in dem die Annahme gemacht wurde, dass stationäre Lösungen existieren, wenn vorgeschriebene Funktionen für Volumen- und Flächenquellen zeitenabhängig sind.

Аннотация—Выведены общие выражения для нестационарного распределения температуры в конечных областях произвольной геометрии при заданном тепловом потоке на всех границах, источниках тепла, зависящих от времени, и произвольных начальных условиях. Источники (или стоки) тепла распределены по всему объёму и могут быть, в частности, поверхностными, линейными или точечными.

Путем введения функций, описывающих некоторые искусственные дополнительные источники, определяются соответствующие псевдостационарные решения, при помощи которых температурные поля выражаются в виде равномерно сходящихся рядов. Общий метод решения применен к детальному изучению задачи в конечном цилиндре очень общего характера, которая ранее не рассматривалась.

Настоящая работа завершает и дополняет предыдущую статью, в которой делалось допущение о существовании стационарных решений при заданных объёмных и поверхностных источниках не зависящих от времени.